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ON A CLASS OF METHODS FOR SOLVING NONLINEAR PROGRAMMING PROBLEMS

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$\boxed{\mathbf{1.}}$ Consider the problem of finding

$$\min_{x\in X} f(x), \qquad X = \{x \in E_n : g(x) = 0\}. \tag{1}$$

Here, $x \in E_n$, E_i is i-dimensional Euclidean space, and the differentiable functions $f(x)$ and $g(x)$ realize the mappings $f : E_n \to E_1$ and $g : E_n \to E_m$, respectively.

In the sequel we assume that there exists a vector $p_* \in E_m$ and a solution of problem (1) such that (x_*, p_*) form a Kuhn–Tucker point, i.e.,

$$f_x(x_*) + g_x(x_*)p_* = 0, \qquad x_* \in X. \tag{2}$$

We introduce a function $\varphi(g^i, p^i)$ of two scalar arguments and the vector

$$\varphi(g, p) = [\varphi(g^1, p^1), \ldots, \varphi(g^m, p^m)].$$

The symbols φ_g, φ_p, φ_{gg}, φ_{pp}, φ_{gp} denote the matrices of first and second derivatives, with the dimensions $m \times 1$, $m \times 1$, m^2, m^2, and m^2, respectively. The matrices of second derivatives are diagonal. A particular element of the matrices is obtained when its scalar arguments are indicated, e.g., $\varphi_g(g^i, p^i)$. We form the generalized Lagrangian

$$H(x, p) = f(x) + \sum_{i=1}^{m} \varphi(g^i(x), p^i) \tag{3}$$

and impose the following requirements on the function φ:

A$_1$. The function $\varphi(g^i, p^i)$ is continuously differentiable, for any real p^i the relation $\varphi_g(0, p^i) = p^i$ holds, and if $g^i \neq 0$, then $\varphi_g(g^i, p^i) \neq p^i$.

Assuming that the problem of unconditional minimization

$$H(x(p), p) = \min_{x\in E_n} H(x, p) \tag{4}$$

has a solution $x = x(p)$, we turn to the problem of finding the roots of the system

$$\Phi(p) = \varphi_g(g(x(p)), p) - p = 0. \tag{5}$$

If p_* is a solution of (5), $x_* = x(p_*)$, and the condition $\mathbf{A}_1$ is satisfied, then (x_*, p_*) is a Kuhn–Tucker point. This allows one to utilize a broad class of methods for finding the roots of systems of equations in order to solve (5) and, in so doing, to solve the problem (1). Assuming that $\Phi(p)$ is differentiable, we obtain the following formula for its derivatives:

$$\Phi_p(p) = [I - \varphi_{gg}(g(x(p)), p) M(x(p), p)]\varphi_{gp}(g(x(p)), p) - I,$$

where

$$\begin{aligned} M(x,p) &= g_x^\top(x) H_{xx}^{-1}(x,p) g_x(x), \\ H_{xx}(x,p) &= L(x,p) + g_x(x)\varphi_{gg}(g(x),p) g_x^\top(x), \\ L(x,p) &= f_{xx}(x) + \sum_{i=1}^{m} g_{xx}^i(x)\varphi_g(g^i(x), p^i). \end{aligned}$$

Here, I is the identity matrix of order m and the superscript $\top$ denotes matrix transposition. We set $x_s = x(p_s)$ and present the following three methods of solving (1):

$$p_{s+1} = \varphi_g(g(x_s), p_s); \tag{6}$$

$$p_{s+1} = p_s - \Phi_p^{-1}(p_s)[\varphi_g(g(x_s), p_s) - p_s]; \tag{7}$$

$$p_{s+1} = \varphi_g(g(x_s), p_s) + Q(x_s, p_s) g(x_s), \quad Q(x,p) = M^{-1}(x,p) - \varphi_{gg}(g(x), p). \tag{8}$$

Here, (6) is, in essence, the method of simple iteration and (7) is an analog of the Newton method.

Let us indicate some examples of simplest functions satisfying $\mathbf{A}_1$:

$$\begin{aligned} \varphi^1(g^i, p^i) &= g^i p^i + (g^i)^2/2, \qquad \varphi^2(g^i, p^i) = g^i(p^i - 1) + \exp(g^i), \\ \varphi^3(g^i, p^i) &= g^i p^i + \frac{1}{2\pi}\left[2g^i \operatorname{arctg} g^i - \ln[1 + (g^i)^2]\right] \exp(-(p^i)^2). \end{aligned}$$

In numerical calculations, it is appropriate to take for $\varphi(g^i, p^i)$ the functions $\varphi^k(\tau g^i, p^i)/\tau$, where $k = 1, 2, 3$, and τ is a positive parameter. For example, taking the first of these functions we obtain that (7) and (8) coincide, and that (6) and (7) lead to the schemes

$$p_{s+1} = p_s + \tau g(x_s), \qquad p_{s+1} = p_s + M^{-1}(x_s, p_s) g(x_s).$$

The first of these methods is well known (see, e.g., [1]).

Let there exist a vector p_* which is a solution of (5). We introduce the equation

$$|\Phi_p(p_*) + I - \lambda I| = 0. \tag{9}$$

The convergence of (6) and (7) follows from the well-known convergence theorems. Let us reformulate these theorems as applied to (5). We denote by G a neighborhood of p_*.

Theorem 1. *Let there exist continuously differentiable functions $x(p)$ and $\Phi(p)$, and let the absolute values of all λ be less than one. Then (6) converges locally to p_*.*

Theorem 2. *Let $H(x,p)$ be twice continuously differentiable in a neighborhood of the Kuhn–Tucker point (x_*, p_*), let the matrix $H_{xx}(x_*, p_*)$ be nonsingular, and let there exist a continuous solution $x(p)$ of problem (4) on G. If $\Phi_p(p_*)$ is nonsingular, and if the matrix $\Phi_p(p)$ satisfies a Lipschitz condition, then (7) converges locally to p_* at quadratic rate. If $M(x_*, p_*)$ is nonsingular, and if the matrix $Q(x(p), p)$ satisfies a Lipschitz condition on G, then (8) converges locally to p_* at quadratic rate.*

In [2], another modification of the Newton method is given, which is suitable for solving (1) and (10) and does not require the auxiliary minimization of H over x. Various modifications of the methods proposed are possible, e.g., the following process is an analog of Seidel's method:

$$p_s^i = \varphi_g(g^i(x(\bar{p}_{si})), p_{s-1}^i), \quad \bar{p}_{si} = [p_s^1, \ldots, p_s^{i-1}, p_{s-1}^i, p_{s-1}^{i+1}, \ldots, p_{s-1}^m], \qquad i \in [1, \ldots, m].$$

$\boxed{\mathbf{2.}}$ The method (6) can be used in the case of problems with inequality-type constraints:

$$\min_{x \in X} f(x), \qquad X = \{x \in E_n : g(x) \leq 0\}. \tag{10}$$

We assume that here also there exists a Kuhn–Tucker point (x_*, p_*), i.e., that at this point (2) holds and $p_*^i \geq 0$, $p_*^i g^i(x_*) = 0$ for $i \in [1, \ldots, m]$. We construct the Lagrangian in the form (3). We denote by $P(g^i)$ the set of real nonnegative solutions of the equation $p^i = \varphi_g(g^i, p^i)$.

Instead of $\mathbf{A}_1$, we impose the following condition on the function φ:

$\mathbf{A}_2$. $P(g^i) = \emptyset$ for $g^i > 0$; $P(g^i) = 0$ for $g^i < 0$; if $a \geq 0$, then $a \in P(0)$; and $\varphi_g(g^i, p^i) \geq 0$ for any g^i and $p^i \geq 0$.

For every p of (4), we find $x = x(p)$, thus defining the system $p = \varphi_g(g(x(p)), p)$. If p_* is a solution of this system, $x_* = x(p_*)$, $p_* \geq 0$, then, by $\mathbf{A}_2$, (x_*, p_*) forms a Kuhn–Tucker point. Taking a $p_0 \geq 0$, we apply the scheme (6) in order to solve (10). Then all $p_s \geq 0$, and the limit points of the sequence $(x(p_s), p_s)$ form the Kuhn–Tucker point.

For example, the following functions satisfy the condition $\mathbf{A}_2$:

$$\begin{aligned}
\varphi^4(g^i, p^i) &= \psi(g_+^i) + p^i e^{g^i}, \\
\varphi^5(g^i, p^i) &= \psi(g_+^i) + p^i \begin{cases} 1 + hg^i + \dfrac{h(h+1)}{2!}(g^i)^2 + \dfrac{h(h+1)(h+2)}{3!}(g^i)^3, & \text{if } g^i \geq 0, \\ \dfrac{1}{(1-g^i)^h}, & \text{if } g^i \leq 0. \end{cases}
\end{aligned}$$

Here, $0 < h$, $g_+^i = \max[0, g^i]$, $\psi(z)$ is a sufficiently smooth function such that $\psi(0) = \psi'(0) = 0$, and $\psi(z) > 0$, $\psi'(z) > 0$, $\psi''(z) > 0$ for $z > 0$ (e.g., $\psi(z) = z^4$).

$\boxed{\mathbf{3.}}$ We denote by q the number of integers belonging to the index set $B = \{j : g^j(x_*) = 0, 1 \leq j \leq m\}$. We introduce the matrices $\bar{\varphi}_{gg}$, $\bar{\varphi}_{pg}$ and $\bar{g}_x$, which coincide with φ_{gg}, φ_{pg} and g_x, respectively, in problem (1). In the case of (10), in the formulas for $\bar{\varphi}_{gg}$, $\bar{\varphi}_{pg}$ and $\bar{g}_x$ only those g^j and their derivatives are retained for which $j \in B$. This means that, in problem (10), the dimensions of $\bar{\varphi}_{gg}$, $\bar{\varphi}_{pg}$ and $\bar{g}_x$ are q^2, q^2, and $n \times q$, respectively.

We present two additional conditions.

$\mathbf{A}_3$. The function φ is such that $\bar{\varphi}_{gg}(g(x_*), p_*)$ is positive definite, $\bar{\varphi}_{pg}(g(x_*), p_*)$ is the identity matrix, and the conditions $\bar{\varphi}_{gg}(g^j(x_*), p_*^i) = 0$, $0 < \varphi_{pg}(g^j(x_*), p_*^j) < 1$ hold in the case (10) for all $j \notin B$.

$\mathbf{A}_4$. The function $H(x, p)$ is twice continuously differentiable in a neighborhood of the Kuhn–Tucker point (x_*, p_*), the columns of the matrix $\bar{g}_x(x_*)$ are linearly independent, and $x^\top L(x_*, p_*)x > 0$ for any nonzero x such that $x^\top \bar{g}_x(x_*) = 0$.

Let μ denote the set of roots of the equation

$$|\bar{\varphi}_{gg}(g(x_*), p_*)\bar{g}_x^\top(x_*)L^{-1}(x_*, p_*)\bar{g}_x(x_*) - \mu I| = 0.$$

Here, the identity matrix I in the case of problem (10) has dimension q^2. If $\mathbf{A}_3$ holds, then the roots of the equation are real; and if they are not all positive, then α denotes the largest negative root.

Theorem 3. *Let there exist a Kuhn–Tucker point at which the matrix L is nonsingular, let $\mathbf{A}_3$ and $\mathbf{A}_4$ hold, let $\mathbf{A}_1$ hold in the case of problem* (1) *and $\mathbf{A}_2$ in the case of problem* (10), *and let the function $\varphi(\tau g, p)/\tau$ be taken for $\varphi(g,p)$. Then, for any $\tau > \bar{\tau}$, the conditions of Theorem* 1 *are satisfied. Moreover, if all $\mu > 0$, then $\bar{\tau} = 0$; otherwise $\bar{\tau} = -2/\alpha$.*

4. If (1) and (10) are convex programming problems, then, instead of (4), one can introduce another auxiliary problem,

$$\min_{x \in E_n} \Gamma(g(x), p, \mu, f(x)), \tag{11}$$

where

$$\Gamma(g(x), p, \mu, f(x)) = \gamma(f(x) - \mu) + \sum_{i=1}^{m} \varphi(g^i(x), p^i)$$

is convex in x, and $\gamma(q)$ is a continuously differentiable function of a scalar argument satisfying the following conditions.

$\mathbf{A}_5$. For all $q \neq 0$, $\gamma(q) > 0$, $\gamma'(q) \neq 0$ and $\gamma(0) = \gamma'(0) = 0$.

Assume that p_s and μ_s are known at the sth step of the iterative process, and that $x_s = x(p_s, \mu_s)$ has been found from (11). We shall construct a method in which

$$p_{s+1} = \varphi_g(g(x_s), p_s) \frac{\gamma'(\bar{f} - \bar{\mu})}{\gamma'(f(x_s) - \mu_s)}. \tag{12}$$

Here, any numbers from the intervals

$$\mu_s \leq \bar{\mu} < \bar{f} \leq f(x_s) + \sum_{i=1}^{m} p_s^i g^i(x_s) = F_s$$

can be taken for $\bar{f}$ and $\bar{\mu}$. We indicate several simple versions of the choice of $\bar{f}$, $\bar{\mu}$, and μ_{s+1}:

$$\begin{array}{lll} \bar{f} = F_s, & \bar{\mu} = \mu_s, & \mu_{s+1} = \mu_s; \\ \bar{f} = F_s, & \bar{\mu} = f(x_s), & \mu_{s+1} = f(x_s); \\ \bar{f} = f(x_s), & \bar{\mu} = \mu_s, & \mu_{s+1} = \mu_s. \end{array}$$

For the algorithms we have presented, it is important that the condition $\Gamma(g(x_*), p_s, \mu_s, f(x)) > 0$ holds. If the initial values μ_0 and p_0 are chosen to satisfy these conditions, then in the case of convex programming problems ($f(x)$ is convex and $g(x)$ is linear in (1) and convex in (10)) this property is automatically preserved in subsequent iterations also. The initial values p_0 and μ_0 can be found, e.g., with the help of the external penalty function method. In this connection, we must have $\mu_0 < f(x_*)$. This condition is satisfied if one sets $\mu_0 \in [f(x_0), F_0]$, where x_0 is the minimum point of the external penalty function.

If we set

$$\Gamma(g(x), p, \mu, f(x)) = (f(x) - \mu)^2 + \sum_{i=1}^{m} \varphi^1(\tau g^i(x), p^i)/\tau \tag{13}$$

in (11) to solve (1), then the method (12) becomes the following:

$$p_{s+1} = (p_s + \tau g(x_s)) \frac{\bar{f} - \bar{\mu}}{f(x_s) - \mu_s}. \tag{14}$$

In the formulas (13) and (14), we can set $p = 0$ and $p_s \equiv 0$, respectively, and any value from the interval $[f(x_s), F_s]$ can be taken for μ_{s+1}. In particular, if $\mu_{s+1} = f(x_s)$, then we arrive at the method of [3]; if $\mu_{s+1} = \mu_s + (\Gamma(g(x_s), 0, \mu_s, f(x_s)))^{1/2} = R_s$, then we obtain the method of [4]. The rate of convergence is greater if we take $\mu_{s+1} = F_s$, since $F_s \geq R_s$. In the case of a linear programming problem, the last algorithm (in which $\mu_{s+1} = F_s$) converges in a finite number of steps.

5. Experience with numerical solution of problems by the method (6), Seidel type methods, and the algorithms of § 4 testifies to their rather high efficiency. The algorithms of § 4, which utilize at every step values of the usual Lagrangian (e.g., $\bar{f} = F_s$, or $\mu_{s+1} = R_s$), require a higher precision in solving the auxiliary problem (11) than that required in the unconditional minimization problem (4) in the methods of § 1 and § 2. It can be shown that the process of solving (4) can be terminated at every sth step as soon as a point x_s is found such that $\|H_x(x_s, p_s)\| \leq e_s$, where $e_s \to 0$ as $s \to \infty$.

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