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Two Parametric Families of LP Problems and Their Applications

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Abstract — A new classification of LP problems is given. Two different parametric families of LP problems are introduced. There are one-to-one correspondences between the solution sets among the problems belonging to the same family. For the construction of necessary and sufficient optimality conditions a combination of the variables of any two problems belonging to different families must be made. An application of new optimality conditions to finding the normal solution of LP problems is given.

1. INTRODUCTION

The main tools in the theory of linear programming (LP) are the duality theory and the Kuhn–Tucker optimality conditions, which involve the variables of primal and dual problems. Sometimes it is useful to transform LP problems and introduce them in different forms. In [1] – [4] various approaches to presentation and classification of LP problems can be found.

In this paper, LP problems are presented in what is called the parametric form, when the objective function and/or the feasible set depend on the values of parameters that belong to some affine sets. Below we propose a nontraditional approach to formulation of linear programming problems. Our considerations are based on using the primal vectors and dual slack vectors as the main variables. Performing linear transformations and changing the objective vectors, we obtain various parametric LP problems, which depend on a choice of one or two parametric vectors belonging to some predefined sets. The parametrization is carried out in such a way that the solution sets should not depend on a specific choice of parameters from the corresponding sets. The relations between the solutions sets and between the optimal values of the objective functions of problems are investigated.

In Section 2, two subclasses of LP problems are distinguished. Namely, primal family (PF) and dual family (DF). All problems from each family are equivalent, though their dimensionalities differ. This means that for the problems belonging to the same family there exist one-to-one correspondences between their feasible sets and solution sets, respectively. Moreover, if we use variables of a problem from one family, we have, to introduce some additional variables from a problem belonging to another family in order to establish necessary and sufficient optimality conditions. There exists a one-to-one correspondence between these families, which is called the conjugacy. Each problem from any family has two problems from another family: dual and conjugate.

Geometric interpretations of these results are given in Section 3. Here we investigate the influence of the parameterized objective vector on the solution set. It is shown that if an LP problem is formulated in the canonical form, then the set of solutions depends only on the projection of the objective vector on the null space of the constraint matrix, which defines the equality constraints.

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In Section 4, we apply our results to obtain the normal (least-norm) solution of the LP problem. The LP problem is reduced to the unconstrained maximization of a smooth concave piecewise quadratic function. The number of variables in this problem is that for the primal LP problem plus one.

2. TWO PARAMETRIC FAMILIES OF LP PROBLEMS AND OPTIMALITY CONDITIONS

Let the primal LP problem be given in the canonical form

$$f_* = \min_{x \in X} c^\top x, \qquad X = \{x \in \mathbb{R}^n : Ax = b, \;\; x \geq 0_n\}. \tag{P}$$

Here and below, A is an $m \times n$ *constraint matrix* of rank $m < n$; $x \in \mathbb{R}^n$ is a primal variable; $c \in \mathbb{R}^n$ is the *objective vector* of (P); $b \in \mathbb{R}^m$ is a constraint vector; and by 0_i we denote the i-dimensional zero vector.

We define the dual problem to (P) as follows:

$$\max_{u \in U} b^\top u, \qquad U = \{u \in \mathbb{R}^m : c - A^\top u \geq 0_n\}. \tag{D}$$

Let X_* and V_* denote the solution sets of the problems (P) and (D), respectively. It is assumed in the sequel that these sets are nonempty. The necessary and sufficient optimality conditions (the Kuhn–Tucker conditions) for the problems (P) and (D) have the form

$$Ax - b = 0_m, \quad c - A^\top u \geq 0_n, \quad D(x)(c - A^\top u) = 0_n, \quad x \geq 0_n. \tag{2.1}$$

Here and below, $D(z)$ denotes the diagonal matrix whose ith diagonal entry is the ith component z^i of the vector z.

For an arbitrary $m \times n$ matrix H introduce the null space (the kernel) of H and the row space of the matrix H (the image of the matrix $H^\top$) denoted by $\ker H$ and $\mathrm{im}\ H^\top$, respectively,

$$\begin{aligned} \ker H &= \{x \in \mathbb{R}^n : Hx = 0_m\}, \\ \mathrm{im}\ H^\top &= \{\xi \in \mathbb{R}^n : \xi = H^\top u, \;\; u \in \mathbb{R}^m\}. \end{aligned}$$

For an arbitrary full rank $m \times n$ matrix H we define its pseudoinverse $n \times m$ matrix H^+, the $n \times n$ matrix $(H^\top)^\parallel$ describes the projection on the row space of H, and the $n \times n$ matrix $(H^\top)^\perp$ describes the projection on the null space of H. If the rank of matrix H is equal to m, then $n \geq m$, the rows of H are linearly independent, and

$$H^+ = H^\top (HH^\top)^{-1}, \quad (H^\top)^\parallel = H^+ H, \quad (H^\top)^\perp = I_n - (H^\top)^\parallel. \tag{2.2}$$

Here and below, I_s denotes the s-dimensional identity matrix.

The dimension of the linear space $\ker A$ is equal to ν, the defect of matrix A. In particular, in the problem (P), we have $\nu = n - m$. The null space and the row space of matrix A are orthogonal complements to each other. The vector space $\mathbb{R}^n$ can be represented as a direct sum of these subspaces, i.e., $\mathbb{R}^n = \mathrm{im}\ A^\top \oplus \ker A$.

In addition to the necessary and sufficient optimality conditions (2.1), we introduce other optimality conditions. For this purpose, following [3, 5, 6] we define a new $\nu \times n$ matrix K. We assume that the rows of K are linearly independent and belong to the null space of matrix A; hence, the subspace $\mathrm{im}\ K^\top$ spanned by them coincides with the null space (kernel) of A. The matrix K can be chosen to be any matrix such that its ν rows form a basis of the null space of A. Thus, im $K^\top$ is the orthogonal complement of the subspace im $A^\top$. Therefore,

$$\text{im } K^\top = \ker A, \quad AK^\top = 0_{m\nu}, \quad \mathbb{R}^n = \text{im } A^\top \oplus \text{im } K^\top, \tag{2.3}$$

where 0_{ij} denotes the $i \times j$ matrix with zero entries.

Matrix K is not uniquely defined; it can be constructed in various ways. If we partition the matrix A as $A = [B \mid N]$, where B is nondegenerate, then we can represent K as $K = [-N^\top (B^{-1})^\top \mid I_\nu]$. If we reduce A by means of Gauss–Jordan transformations to the form $A = [I_m \mid N]$, then we can represent K as $K = [-N^\top \mid I_\nu]$.

In some special cases, the construction of the matrix K is extremely simple. For instance, suppose that an LP problem involves inequality constraints $Nz \geq b$, where N is an $m \times \nu$ matrix and $z \in \mathbb{R}^\nu_+$. Introducing auxiliary variables $\xi \in \mathbb{R}^m_+$, we represent the vector $x \in \mathbb{R}^n$ as the union of the vectors z and ξ, i.e., $x^\top = [z^\top, \xi^\top]$. The feasible set can be written in the same form as for problem (P), where $A = [N \mid -I_m]$. Therefore, we can define K as $K = [I_\nu \mid N^\top]$.

We define a vector $d \in \mathbb{R}^\nu$ and dual slack vector $v \in \mathbb{R}^n$ by the formulas $d = Kc$ and

$$v = c - A^\top u. \tag{2.4}$$

Let us introduce two affine sets

$$\bar{X} = \{x \in \mathbb{R}^n : Ax = b\}, \qquad \bar{V} = \{v \in \mathbb{R}^n : Kv = d\}.$$

Here and below, $\bar{x}$ and $\bar{v}$ denote arbitrary fixed n-dimensional vectors satisfying the conditions $\bar{x} \in \bar{X}$ and $\bar{v} \in \bar{V}$. Note that some components of $\bar{x}$ and $\bar{v}$ may be negative. In the simplest case, one can take $\bar{v} = c$. By virtue of the fact that the rank of the matrix A is equal to m and $n > m$, we always have $\bar{X} \neq \emptyset$ and $\bar{V} \neq \emptyset$.

A vector $\bar{v} \in \bar{V}$ can easily be found. It is enough to take an arbitrary vector $\bar{u} \in \mathbb{R}^m$ and define vector $\bar{v}$ as

$$\bar{v} = c - A^\top \bar{u}. \tag{2.5}$$

Taking into account (2.3) and (2.4), we come to conclusion that $\bar{v} \in \bar{V}$.

Substituting $c = \bar{v} + A^\top \bar{u}$ into the objective function of the problem (P), we get

$$c^\top x = \bar{v}^\top x + b^\top u(\bar{v}) \tag{2.6}$$

for any $x \in \bar{X}$. Therefore, we can replace the original problem (P) by the following modified problem:

$$f^1_*(\bar{v}) = \min_{x \in X} \bar{v}^\top x, \qquad X = \{x \in \mathbb{R}^n : Ax = b, \ \ x \geq 0_n\}. \tag{P$_x$}$$

The problems (P) and (P$_x$) are quite similar, differing slightly only in the choice of the objective vector. In the problem (P) this vector is defined in the description of the problem, and in (P$_x$) we can choose any vector $\bar{v} \in \bar{V}$. Hence, in the second case we actually have a variety of problems. So, we can refer to (P$_x$) as a *one-parametric problem.*

The solution sets of the problems (P) and (P$_x$) coincide and are independent of the specific choice of the vector $\bar{v} \in \bar{V}$ or, due to (2.5), of arbitrary vector $\bar{u} \in \mathbb{R}^m$. The optimum values of the objective functions differ by a constant depending on the specific choice of $\bar{u} \in \mathbb{R}^m$. The difference is equal to zero if $\bar{u} = 0_m$, because in this case $\bar{v} = c$ and $f_* = f^1_*(c)$. So, the problem (P) is a particular case of (P$_x$); therefore we can exclude (P) from consideration.

By making a change of variables in the problem (P$_x$) given in the canonical form, we will obtain the problem in a standard form, where only the inequality constraints define the feasible set. The general solution of the nonhomogeneous system of linear equations $Ax = b$ can be written in the form

$$x = \bar{x} - K^\top y, \tag{2.7}$$

where $\bar{x}$ is a particular solution of the system, $K^\top y$ is the general solution of the homogeneous system $Ax = 0_m$, and $y \in \mathbb{R}^\nu$. Let us define the set

$$Y = \{y \in \mathbb{R}^\nu : \bar{x} - K^\top y \geq 0_n\}. \tag{2.8}$$

Formula (2.7) can be considered as a linear transformation from the linear vector space $\mathbb{R}^\nu$ to another linear vector space $\mathbb{R}^n$. If $y \in Y$, then from (2.7), (2.3) and (2.8) we can get $Ax = A\bar{x} = b$, $x \geq 0_n$. Hence, the transformation (2.7) maps the set Y to the feasible set X of the problems (P) and (P$_x$). We refer to X as the image of Y under the space transformation (2.7). There exists a one-to-one correspondence between X and Y. Indeed, for each $y \in Y$ one can uniquely determine $x \in X$ by formula (2.7). For overdetermined system (2.7) containing n linear equations and ν unknowns y, the pseudosolution

$$y(x) = (KK^\top)^{-1}K(\bar{x} - x) = (K^\top)^+(\bar{x} - x) \tag{2.9}$$

always exists. It solves (2.7) and is unique if and only if $\bar{x} - x \in \operatorname{im} K^\top$. This inclusion holds if and only if $x \in \bar{X}$. Thus, for any $x \in \bar{X}$, formula (2.9) determines an affine transformation that is the inverse of (2.7). Therefore, one can write

$$Y = (K^\top)^+(\bar{x} - X). \tag{2.10}$$

Let us express the objective function of the problem (P$_x$) in terms of the variable y. Substituting (2.7) into the objective function of the problem (P$_x$) and using the equality $\bar{v}^\top K^\top = c^\top K^\top = d^\top$, we obtain

$$\bar{v}^\top x = \bar{v}^\top \bar{x} - \bar{v}^\top K^\top y = \bar{v}^\top \bar{x} - d^\top y. \tag{2.11}$$

Taking into account (2.11) and the fact that X is the image of Y, one can write the problem (P$_x$) in the standard form, where the feasible set Y is the intersection of n half-spaces:

$$\max_{y \in Y} d^\top y, \qquad Y = \{y \in \mathbb{R}^\nu : \bar{x} - K^\top y \geq 0_n\}. \tag{P$_y$}$$

Thus, m equality constraints are excluded from the definition of the set X, and we deal with inequality constraints only. We can consider (2.7) as a linear change of variables. This formula relates the new variable y to the old variable x.

There exist one-to-one correspondences defined by (2.7) and (2.9) between the feasible sets X and Y, as well as between the solution sets X_* and Y_* of the problems (P$_x$) and (P$_y$), respectively. In particular, for the solution sets X_* and Y_* these correspondences can be written in the form

$$X_* = \bar{x} - K^\top Y_*, \qquad Y_* = (K^\top)^+(\bar{x} - X_*). \tag{2.12}$$

Let us introduce a new variable

$$g = K^\top y. \tag{2.13}$$

From the condition $\bar{v} \in \bar{V}$ we obtain $d^\top = \bar{v}^\top K^\top$; therefore, due to (2.7) we have two expressions for the objective function of the problem (P$_y$):

$$d^\top y = \bar{v}^\top(\bar{x} - x), \qquad d^\top y = \bar{v}^\top g. \tag{2.14}$$

On substituting the first representation in the objective function of the problem (P_y) and transforming the feasible set Y, we obtain another problem,

$$\min_{[x,y]\in Z} \bar{v}^\top x, \qquad Z = \{[x,y] : K^\top y + x = \bar{x}, \;\; x \geq 0_n\}. \tag{P_{xy}}$$

The solution set of this problem is $Z_* = [X_*, Y_*]$.

From definition (2.13) it follows that $g \in \mathrm{im}\, K^\top = \ker A$. Therefore, we can define g as a solution of the linear homogeneous system $Ag = 0_n$. Using the second representation given by (2.14) and formula (2.7), we obtain from (P_y) another problem,

$$\max_{g\in G} \bar{v}^\top g, \qquad G = \{g \in \mathbb{R}^n : Ag = 0_m, \;\; g \leq \bar{x}\}. \tag{P_g}$$

Thereby, the use of variable y in the problem (P_{xy}) is excluded and the number of variables is diminished. The solution set G_* of the problem (P_g) is connected with Y_* and X_* by relations

$$G_* = K^\top Y_* = \bar{x} - X_*. \tag{2.15}$$

Actually, the problems (P_x), (P_y), (P_{xy}), and (P_g) can be considered as the same problem. The difference in the formulation is implied by the change of variables (2.7) and (2.13), which allow us to write the objective functions in various forms and transform the feasible sets. One can say that (P), (P_x), (P_y), (P_{xy}), and (P_g) are equivalent problems in the sense that the solution sets of (P) and (P_x) coincide and there exists a one-to-one correspondence between X_* and other solution sets. We will refer to the problems (P), (P_x), (P_y), (P_{xy}), and (P_g) as *a primal family* of linear programming problems. Now by analogy we introduce *a dual family* of problems. We change variables in the problem (D), setting

$$u = \bar{u} + w, \tag{2.16}$$

where $\bar{u}$ was introduced in (2.5). Substituting (2.16) in (D) and in the definition of U, we obtain a new problem in the standard form:

$$\max_{w\in W} b^\top w, \qquad W = \{w \in \mathbb{R}^m : \bar{v} - A^\top w \geq 0_n\}. \tag{D_w}$$

Simultaneously we get the following relation between the optimal values of the corresponding objective functions

$$\max_{w\in W} b^\top w = \max_{u\in U} b^\top u - b^\top \bar{u}. \tag{2.17}$$

We denote by W_* the solution set of the problem (D_w). There exists a one-to-one correspondence between the feasible sets U and W defined as $U = \bar{u} + W$. A similar relationship $U_* = \bar{u} + W_*$ holds between the solution sets.

Let us introduce the following relation:

$$v = \bar{v} - A^\top w, \tag{2.18}$$

where $\bar{v} \in \bar{V}$, and define a set

$$V = \{v \in \mathbb{R}^n : Kv = d, \;\; v \geq 0_n\}.$$

In (2.18), the vector w can be regarded as an implicit function of a dual slack vector v of larger dimension. System (2.18) is overdetermined, that is, it is solvable with respect to the vector w not for all vectors v and $\bar{v}$. However, the pseudosolution

$$w(v) = (AA^\top)^{-1}A(\bar{v} - v) = (A^\top)^+(\bar{v} - v) \tag{2.19}$$

always exists and is unique. This pseudosolution is also a unique solution of system (2.18) if and only if the vector $\bar{v}-v \in \operatorname{im} A^\top$. In this case, according to (2.3), matrix K is orthogonal to the vector $\bar{v}-v$, i.e., $Kv = K\bar{v} = d$. Since $\bar{v} \in \bar{V}$, one can assert that system (2.18) is uniquely solvable with respect to w if and only if $v \in \bar{V}$. If we add the condition $w \in W$, then relation (2.18) implies that the corresponding vector $v \geq 0_n$ and, hence, $v \in V$. Therefore, there exists a one-to-one correspondence between the feasible set W of the problem (D_w) and the set V.

We represent the objective function of the problem (D_w) in terms of the dual slack vector v. For this purpose, we substitute $b = A\bar{x}$ into the objective function of the problem (D_w) and, taking into account (2.18), we obtain

$$b^\top w = \bar{x}^\top A^\top w = \bar{x}^\top \bar{v} - \bar{x}^\top v. \tag{2.20}$$

Thus, the problem (D_w) is reduced to the LP problem written in the canonical form

$$f_*^2(\bar{v}) = \min_{v \in V} \bar{x}^\top v, \qquad V = \{v \in \mathbb{R}^n : Kv = d,\ v \geq 0_n\}. \tag{D_v}$$

Let V_* denote the solution set of this problem. The relations between the feasible sets V and W and between the solution sets V_* and W_* can be expressed as follows:

$$\begin{array}{rclcrcl} V & = & \bar{v} - A^\top W, & \qquad & W & = & (A^\top)^+(\bar{v} - V), \\ V_* & = & \bar{v} - A^\top W_*, & \qquad & W_* & = & (A^\top)^+(\bar{v} - V_*). \end{array} \tag{2.21}$$

Any particular problem from (D_v) has one and the same solution set V_*, which is independent of the specific choice of the vector $\bar{x} \in \bar{X}$. Formulas (2.21) define a one-to-one correspondence between the solution sets V_* and W_* of the problems (D_v) and (D_w), respectively. Let a vector $q \in \mathbb{R}^n$ be defined as

$$q = A^\top w. \tag{2.22}$$

Since $\bar{x} \in \bar{X}$, it follows that

$$b^\top w = \bar{x}^\top A^\top w = \bar{x}^\top q. \tag{2.23}$$

Multiplying (2.22) from the left by K and taking into account orthogonality condition in (2.3), we obtain $Kq = 0_\nu$. From the feasibility condition of the problem (D_w) it follows that $q \leq \bar{v}$. So, the problem (D_w) can be reformulated as

$$\max_{x \in Q} \bar{x}^\top q, \qquad Q = \{q \in \mathbb{R}^n : Kq = 0_\nu,\ q \leq \bar{v}\}, \tag{D_q}$$

where the solution set has the form

$$Q_* = A^\top W_* = \bar{v} - V_*. \tag{2.24}$$

Since the rank of A is m, we have a one-to-one correspondence between Q_* and W_*.

From (2.18) and (2.23) we obtain $b^\top w = \bar{x}^\top(\bar{v} - v)$. So, we come to conclusion that the problem

$$\min_{[v,w] \in T} \bar{x}^\top v, \qquad T = \{[v, w] : A^\top w + v = \bar{v},\ v \geq 0_n\} \tag{D_{vw}}$$

is equivalent to the problem (D_w). The solution set of (D_{vw}) is $T_* = [V_*, W_*]$. The problems (D), (D_w), (D_v), (D_q), and (D_{vw}) belong to the *dual family.* All these problems can be regarded as equivalent.

The problems (P_x), (P_y), (D_w), and (D_v) depend on a parameter $\bar{x}$ or $\bar{v}$. Therefore, we refer to them as *one-parametric* problems. The problems (P_{xy}), (P_g), (D_q), and (D_{vw}) depend simultaneously on two parameters $\bar{x}$ and $\bar{v}$, and we call them *two-parametric problems.*

The class of these eight problems is rather large. In particular, it contains the original problems (P) and (D). Indeed, if we substitute the vector c for $\bar{v}$ in the problems (P_x) and (D_w), then we obtain the problems (P) and (D)), respectively. Therefore, we can exclude (P) and (D) from consideration if we deal with (P_x) and (D_w).

There exists a symmetry between the problems (P_x) and (D_v). Both problems are of the same type, and the objective functions and the feasible sets are similar. The difference consists only in the specific data which define the corresponding matrices, vectors, and numbers. The problem (D_v) is called *conjugate* to (P_x). Similar symmetry exists between the problems (P_y) and (D_w), between (P_{xy}) and (D_{vw}), and between (P_g) and (D_q). Therefore, each of these pairs can be called a pair of *mutually conjugate* problems.

We express this correspondence between the problems, variables, matrices, vectors, numbers which describe these problems, feasible sets, and solution sets by the symbol $\Longleftrightarrow$, i.e.,

$$(\mathrm{P}_x) \Leftrightarrow (\mathrm{D}_v), \qquad (\mathrm{P}_y) \Leftrightarrow (\mathrm{D}_w), \qquad (\mathrm{P}_{xy}) \Leftrightarrow (\mathrm{D}_{vw}), \qquad (\mathrm{P}_g) \Leftrightarrow (\mathrm{D}_q),$$

$$A \Leftrightarrow K, \quad X \Leftrightarrow V, \quad W \Leftrightarrow Y, \quad X_* \Leftrightarrow V_*, \quad W_* \Leftrightarrow Y_*, \quad G_* \Leftrightarrow Q_*, \quad Z_* \Leftrightarrow T_*,$$

$$x \Leftrightarrow v, \qquad b \Leftrightarrow d, \qquad m \Leftrightarrow \nu, \qquad \bar{v} \Leftrightarrow \bar{x}, \qquad y \Leftrightarrow w, \qquad g \Leftrightarrow q.$$

The following figures summarize the relationships between the different problems.

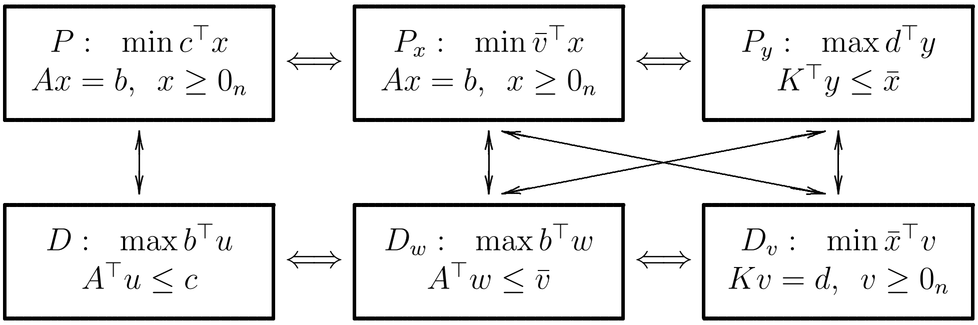

Fig. 1. Original and one-parametric problems.

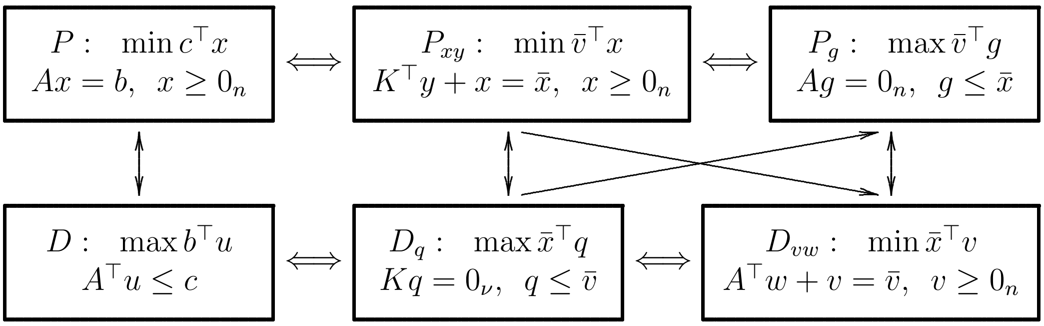

Fig. 2. Original and two-parametric problems.

The notation $B \Leftrightarrow C$ means that the problems (B) and (C) are equivalent, the vertical arrows between the problems (B) and (C) denote their mutual duality, and the diagonal arrows mean the mutual conjugacy.

Below we present some results concerning the properties of all introduced problems.

Theorem 1 (an analogue of weak duality). *Assume that all the variables belong to the feasible sets of the corresponding problems. Then for one-parametric problems the following inequalities hold:*

$$\bar{v}^\top x + \bar{x}^\top v \ge \bar{x}^\top \bar{v} \ge d^\top y + b^\top w, \tag{2.25}$$

and for two-parametric problems the following similar inequalities valid:

$$\bar{v}^\top x + \bar{x}^\top v \geq \bar{x}^\top \bar{v} \geq \bar{v}^\top g + \bar{x}^\top q. \tag{2.26}$$

Proof. Multiplying (2.18) by $x \in X$, we obtain

$$v^\top x = \bar{v}^\top x - w^\top Ax = \bar{v}^\top x - w^\top A\bar{x} = \bar{v}^\top x + \bar{x}^\top (v - \bar{v}) \geq 0, \qquad \bar{v}^\top x + \bar{x}^\top v \geq \bar{x}^\top \bar{v}.$$

Multiplying (2.7) by $v \in V$, we get

$$v^\top x = \bar{x}^\top (\bar{v} - Aw) - y^\top Kv = \bar{x}^\top (\bar{v} - Aw) - y^\top K\bar{v} = \bar{x}^\top \bar{v} - b^\top w - d^\top y \geq 0, \quad \bar{x}^\top \bar{v} \geq b^\top w + d^\top y.$$

Thus, (2.25) is proved. Taking into account (2.14) and (2.23), we have (2.26). □

From the left inequality in (2.25) and equality (2.18), we obtain $\bar{v}^\top x \geq \bar{x}^\top (\bar{v} - v) = \bar{x}^\top A^\top w = b^\top w$, where $x \in X$, $w \in W$. If $\bar{v} = c$, then $\bar{u} = 0_m$ and $W = U$, and we conclude that for the problems (P) and (D) the well-known weak duality inequality $c^\top x \geq b^\top u$ holds for all $x \in X$, $u \in U$.

By opt Q we denote the optimal value of the objective function of the problem (Q). The optimal values of all objective functions, except the problems (P) and (D), depend either on one of the parameters $\bar{v}$ and $\bar{x}$ or on both. For example, we write opt $\mathrm{P}_x(\bar{v})$, opt $\mathrm{D}_q(\bar{v}, \bar{x})$, opt $\mathrm{D}_w(\bar{v})$, and opt $\mathrm{P}_y(\bar{x})$.

Theorem 2 (an analogue of duality). *If there exists a solution of at least one of the ten LP problems under consideration, then there exist solutions of the nine other problems. The optimal values of the objective functions of these problems are related to each other as follows:*

$$\text{opt } P = \text{opt } P_x(\bar{v}) + b^\top \bar{u} = \text{opt } D = \text{opt } D_w(\bar{v}) + b^\top \bar{u}, \tag{2.27}$$

$$\begin{aligned} \text{opt } P_x(\bar{v}) + \text{opt } D_v(\bar{x}) &= \text{opt } P_y(\bar{x}) + \text{opt } D_w(\bar{v}) = \\ &= \text{opt } P_x(\bar{v}) + \text{opt } P_y(\bar{x}) = \\ &= \text{opt } D_v(\bar{x}) + \text{opt } D_w(\bar{v}) = \bar{x}^\top \bar{v}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} \text{opt } P_{xy}(\bar{v}, \bar{x}) + \text{opt } D_{wv}(\bar{v}, \bar{x}) &= \text{opt } P_g(\bar{v}, \bar{x}) + \text{opt } D_q(\bar{v}, \bar{x}) = \\ &= \text{opt } P_{xy}(\bar{v}, \bar{x}) + \text{opt } P_g(\bar{v}, \bar{x}) = \\ &= \text{opt } D_q(\bar{v}, \bar{x}) + \text{opt } D_{wv}(\bar{v}, \bar{x}) = \bar{x}^\top \bar{v}. \end{aligned} \tag{2.29}$$

Proof. The existence of the solutions follows from the equivalence of the problems in the PF and DF and from the ordinary duality theory applied to the corresponding problems in the PF and DF. Formulas (2.6) and (2.17) imply (2.27).

In accordance with the LP duality theorem, the objective functions calculated for the solutions of mutually dual problems are equal, which can be written in the form

$$\text{opt P} = \text{opt D} = f_*, \qquad \text{opt } \mathrm{P}_x(\bar{v}) = \text{opt } \mathrm{D}_w(\bar{v}), \qquad \text{opt } \mathrm{P}_y(\bar{x}) = \text{opt } \mathrm{D}_v(\bar{x}). \tag{2.30}$$

Therefore, (2.28) follows from relations (2.11), (2.20), and (2.30). The statement about properties of two-parametric problems (2.29) can be proved in a similar way. □

We say that x and v satisfy the complementary slackness condition (CSC) if $x^i v^i = 0$ for all $1 \leq i \leq n$.

Theorem 3. *Let the vectors x and v be feasible for the problems* (P_x) *and* (D_v)*, respectively. Then the vectors x and v satisfy CSC if and only if the equality*

$$\bar{v}^\top x + \bar{x}^\top v = \bar{x}^\top \bar{v} \tag{2.31}$$

holds.

Proof. The conditions $x \in X \subset \bar{X}$ and $v \in V \subset \bar{V}$ imply that $\bar{x} - x \in \ker A = \mathrm{im}\, K^\top$ and $\bar{v} - v \in \ker K = \mathrm{im}\, A^\top$. From the orthogonality of the vectors $\bar{x} - x$ and $\bar{v} - v$, we obtain

$$(\bar{x} - x)^\top (\bar{v} - v) = \bar{x}^\top \bar{v} - \bar{v}^\top x - \bar{x}^\top v + x^\top v = 0. \tag{2.32}$$

For $x \geq 0_n$ and $v \geq 0_n$, the equality $x^\top v = 0$ is equivalent to the CSC. Therefore, (2.32) implies that, for feasible x and v, the CSC holds if and only if (2.31) is valid. $\square$

Corollary 1. *For any x and $\bar{x}$ from $\bar{X}$ and any v and $\bar{v}$ from $\bar{V}$, formula* (2.32) *is valid.*

Corollary 2. *For any x_* and v_* that are solutions of the problems* (P_x) *and* (D_v)*, respectively, the equality $x_*^\top v_* = 0$ holds.*

This corollary can be obtained from (2.31) by setting x and $\bar{x}$ equal to x_* and v and $\bar{v}$ equal to v_*.

Corollary 3. *The variables x, u, v, w, y, g, and q are optimal for their respective problems if and only if they satisfy at least one of the following conditions:*

$$\begin{array}{llll}
Ax = b,\ x \geq 0_n, & Kv = d,\ \ v \geq 0_n, & \bar{v}^\top x + \bar{x}^\top v = \bar{x}^\top \bar{v}, & (P_x\&D_v) \\
\bar{x} - K^\top y \geq 0_n, & \bar{v} - A^\top w \geq 0_n, & d^\top y + b^\top w = \bar{x}^\top \bar{v}, & (P_y\&D_w) \\
Ax = b,\ x \geq 0_n, & \bar{v} - A^\top w \geq 0_n, & \bar{v}^\top x - b^\top w = 0, & (P_x\&D_w) \\
\bar{x} - K^\top y \geq 0_n, & Kv = d,\ v \geq 0_n, & \bar{x}^\top v - d^\top y = 0, & (P_y\&D_v) \\
K^\top y + x = \bar{x},\ x \geq 0_n, & A^\top w + v = \bar{v},\ v \geq 0_n, & \bar{v}^\top x + \bar{x}^\top v = \bar{x}^\top \bar{v}, & (P_{xy}\&D_{vw}) \\
Ag = 0_m,\ g \leq \bar{x}, & Kq = 0_\nu,\ q \leq \bar{v}, & \bar{v}^\top g + \bar{x}^\top q = \bar{x}^\top \bar{v}, & (P_g\&D_q) \\
Ax = b,\ x \geq 0_n, & Kq = 0_\nu,\ q \leq \bar{v}, & \bar{v}^\top x - \bar{x}^\top q = 0, & (P_x\&D_q) \\
\bar{x} - K^\top y \geq 0_n, & A^\top w + v = \bar{v},\ v \geq 0_n, & d^\top y - \bar{x}^\top v = 0. & (P_y\&D_{vw})
\end{array}$$

In the first column the feasible sets of the corresponding problems PF are given. In the second column we indicate the feasible sets of the problems which belong to DF. In the third column the relations between optimal values of objective functions are given (in accordance with Theorem 2). In the last column we indicate the problems whose variables are used in optimality conditions. In fact, we can combine each problem from the PF with any problem from the DF and obtain 16 optimality conditions. Here, for brevity, we present only eight optimality conditions.

We clarify the assertions of Corollary 3 in terms of variables that are used in conditions (P_x&D_v). If x and v satisfy these conditions, then $x \in X_*$ and $v \in V_*$ (the sufficient conditions of an extremum). If $x \in X_*$ and $v \in V_*$, then x and v satisfy (P_x& D_v) (the necessary conditions of an extremum). The last assertion can be strengthened. Following the standard reasoning used in the theory of LP [7], one can show that if $x \in X_*$, then there exists a vector v such that (P_x&D_v) holds. Similarly, if $v \in V_*$, then there exists x such that conditions (P_x&D_v) are satisfied. If $x \in X_*$, then from (2.12) and (2.15) we obtain $y \in Y_*$, and $g \in G_*$. Similarly, if $v \in V_*$, then from (2.21) and (2.24) we get $w \in W_*$ and $q \in Q_*$.

Note that each of the optimality conditions (P_x&D_v)–(P_y& D_{vw}) is distinguished by the number of variables and constraints. The optimality conditions (P_y&D_w) have the smallest number of unknowns (n) and constraints ($2n + 1$).

Conditions (P_x&D_w) and (P_y&D_v) are the well-known Kuhn–Tucker conditions for pairs of the mutually dual problems (P_x), (D_w) and (D_v), (P_y), where the CSC are replaced by equivalent conditions of equality of the objective functions of the appropriate problems. Conditions (P_x&D_v) and (P_y&D_w) are the optimality conditions for the mutually conjugate problems (P_x), (D_v) and the problems (P_y), (D_w).

We mention one important case of choosing parametric vectors $\bar{x}$ and $\bar{v}$. Let be $\bar{x} = \bar{v}$. We denote this vector by ξ. It satisfies two conditions, $A\xi = b$ and $K\xi = d$, and is a unique intersection point of two affine sets. Both the matrices A and K have maximal ranks. Therefore, vector ξ is uniquely defined for each LP problem: $\xi = M^{-1}h$, where $(n \times n)$-matrix $M^\top = [A^\top | K^\top]$ has rank n and $h^\top = [b^\top, d^\top]$ is an n-dimensional vector. In all previous formulas in this case we can write $\bar{x}^\top \bar{v} = \|\xi\|^2$. For example, formulas (2.25) and (2.26) from Theorem 1 can be expressed as

$$\xi^\top(x+v) \geq \|\xi\|^2 \geq d^\top y + b^\top w, \qquad \xi^\top(x+v) \geq \|\xi\|^2 \geq \xi^\top(g+q).$$

The last condition from (P_x&D_v) can be rewritten as follows:

$$\xi^\top(x+v) = \|\xi\|^2 \tag{2.33}$$

In other words, the vector ξ is orthogonal to the vector $x + v - \xi$ and the projection of $x + v$ on ξ coincides with ξ.

Theorem 4. *Let the vectors x and v be feasible for the problems* (P_x) *and* (D_v)*, respectively. Then the projection of vector $x + v$ on vector ξ coincides with this vector if and only if x and v are solutions of* (P_x) *and* (D_v)*, respectively.*

In the end of this section, we present two mutually conjugate problems:

$$f_*^1(\xi) = \min_{x \in X} \xi^\top x, \qquad X = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0_n\}, \tag{P_x}$$

$$f_*^2(\xi) = \min_{v \in V} \xi^\top v, \qquad V = \{v \in \mathbb{R}^n : Kv = d, \ v \geq 0_n\}. \tag{D_v}$$

Both problems have the same objective vector ξ. If x and v are feasible, then the necessary and sufficient optimality condition is the orthogonality of these vectors, which is equivalent to linear condition (2.33).

3. GEOMETRIC INTERPRETATION

We use the superscripts $^\|$ and $^\perp$ at n-dimensional vectors to denote the orthogonal projections of these vectors on the row subspace of the matrix A and into the kernel of the matrix A, respectively. For example, $x^\| = (A^\top)^\| x$, $x^\perp = (A^\top)^\perp x$, where $(A^\top)^\|$ and $(A^\top)^\perp$ are defined in accordance with formulas (2.2):

$$(A^\top)^\| = A^+ A = A^\top (AA^\top)^{-1} A, \qquad (A^\top)^\perp = I_n - (A^\top)^\|. \tag{3.1}$$

By a direct substitution, we verify that for any vectors $\bar{x} \in \bar{X}$ and $\bar{v} \in \bar{V}$ the following relations hold:

$$A\bar{x}^\perp = 0_m, \qquad K\bar{v}^\| = 0_\nu, \qquad A\bar{x}^\| = A\bar{x} = b, \qquad K\bar{v}^\perp = K\bar{v} = d.$$

By $\tilde{x}$ and $\tilde{v}$ we denote the normal solutions (solutions with the minimal Euclidean norm) of the systems $Ax = b$, $Kv = d$. These solutions can be represented in the following different but equivalent forms:

$$\tilde{x} = A^+ b = A^\top (AA^\top)^{-1} b = (A^\top)^\| \bar{x} = \bar{x}^\|, \tag{3.2}$$

$$\tilde{v} = K^+ d = K^\top (KK^\top)^{-1} d = (K^\top)^\| \bar{v} = (A^\top)^\perp \bar{v} = \bar{v}^\perp. \tag{3.3}$$

Let $\bar{x}$ and $\bar{v}$ be normal solutions of the corresponding linear systems, i.e., $\bar{x} = \tilde{x}$ and $\bar{v} = \tilde{v}$. In this case we have $\bar{x}^\top \bar{v} = 0$, since $\tilde{x} \in \text{im } A^\top$ and $\tilde{v} \in \text{im } K^\top$. So, if all components of the normal solutions of the systems $Ax = b$ and $Kv = d$ are simultaneously nonnegative, then these normal solutions are solutions of the problems (P_x) and (D_v), respectively.

Both parameters $\bar{x} \in \bar{X}$ and $\bar{v} \in \bar{V}$ can be changed when finding solutions of the considered problems. If the pair $[\bar{x}, \bar{v}]$ tends to $[x_*, v_*]$, then the optimal values of objective functions in (P_x) and (D_v) tend to zero.

Note that the vector c is absent in all optimality conditions (P_x& D_v),...,(P_y&D_{vw}). Only vector d depends on the choice of c. More precisely, it depends only on $c^\perp$, i.e., the projection of c on the null space of the matrix A. Indeed, from (2.4) in view of (2.3), we have

$$K(c^\perp + c^\parallel) = Kc^\perp + K(A^\top)^\parallel c = Kc^\perp = d.$$

On the other hand, the optimal value of the objective function $c^\top x_*$ depends on the choice of $c^\parallel$, since, for $x_* \in X_*$, we have $c^\top x_* = (c^\perp + c^\parallel)^\top x_*$. According to (2.27), the optimal values of the objective functions differ by the quantity $b^\top \bar{u}$. Using (3.2) and the formula $\bar{u} = (AA^\top)^{-1}A(c - \bar{v})$, we obtain

$$\begin{aligned} b^\top \bar{u} &= b^\top (AA^\top)^{-1} A(c - \bar{v}) = \tilde{x}^\top (c - \bar{v}) = (\bar{x}^\parallel)^\top (c - \bar{v}) = (\bar{x}^\parallel)^\top (c - \bar{v})^\parallel, \\ f_* - f_*^1(\bar{v}) &= x_*^\top (c - \bar{v}) = \tilde{x}^\top (c - \bar{v}). \end{aligned}$$

Let $h, p \in \mathbb{R}^n$. Making use of these vectors, we change vectors $\bar{v}$, $\bar{x}$ in the problems (P_x), (D_v) and obtain two perturbed problems,

$$\min_{x \in X} (\bar{v} + h)^\top x, \qquad (\mathrm{P}'_x)$$

$$\min_{v \in V} (\bar{x} + p)^\top v. \qquad (\mathrm{D}'_v)$$

Theorem 5. *Let $h \in \text{im } A^\top$ and $p \in \ker A$. Then the solution set X_* of the problem* (P_x) *coincides with the solution set of the problems* (P'_x), *and the solution set V_* of the problem* (D_v) *coincides with the solution set of the problem* (D'_v).

Proof. Since $h \in \text{im } A^\top = \ker K$, we obtain $K(\bar{v} + h) = d$, i.e., $\bar{v} + h \in \bar{V}$. Therefore, the problem (P'_x) belongs to the family (P_x), in which, by construction, all the problems have the same solution set and the optimal value of the objective functions of any two problems differs only by a constant. Since $p \in \ker A = \text{im } K^\top$, the symmetry of the problems (D_v) and (P_x) implies the second assertion of the theorem. □

If the objective vectors c, $\bar{v}$, and $\bar{x}$ of the problems (P), (P_x), and (D_v) are replaced by $c - c^\parallel$, $\bar{v} - \bar{v}^\parallel$, and $\bar{x} - \bar{x}^\perp$, respectively, and if we take into account the inclusions $c^\parallel, \bar{v}^\parallel \in \text{im } A^\top$ and $\bar{x}^\perp \in \ker A$, then by Theorem 5 we obtain the following result.

Theorem 6. *The solution sets of the problems* P, P_x, *and* D_v *do not depend on the projections* $c^\parallel$, $\bar{v}^\parallel$, *and* $\bar{x}^\perp$, *respectively.*

Thus, instead of the vectors c and $\bar{v}$ involved in the objective function of the problems (P) and (P_x), one can take arbitrary vectors whose projections on the kernel of the matrix A are equal to $c^\perp$ and $\bar{v}^\perp$. In this case, the solution sets of the problems (P) and (P_x) coincide. Similarly, in the problem (D_v), one can take any vector whose projection on im $A^\top$ is equal to $\bar{x}^\parallel$. In this case, the set V_* is not changed.

The projections $x_*^{\|}$ and $v_*^{\perp}$ of the optimal vectors x_* and v_* in the problems (P), (P_x), and (D_v) are determined directly by formulas (3.2), (3.3). Indeed, taking into account the conditions $x_* \in \bar{X}$ and $v_* \in \bar{V}$, we obtain

$$x_*^{\|} = \tilde{x} = \tilde{x}^{\|} = \bar{x}^{\|}, \qquad v_*^{\perp} = \tilde{v} = \tilde{v}^{\perp} = \bar{v}^{\perp} = c^{\perp}. \tag{3.4}$$

Therefore, solving the problems (P), (P_x), and (D_v) reduces to finding two orthogonal vectors $x_*^{\perp} \in \ker A$ and $v_*^{\|} \in \mathrm{im}\ A^{\top}$ such that $x_* = \bar{x} + x_*^{\perp} \geq 0_n$, $v_* = \bar{v} + v_*^{\|} \geq 0_n$, and $x_*^{\top} v_* = 0$. The desired vectors are representable in the form $x_*^{\top} = -K^{\top} y$, $v_*^{\|} = -A^{\top} w$, where $y \in \mathbb{R}^{\nu}$, $w \in \mathbb{R}^m$. Thus, vectors y and w satisfying (P_y&D_w) are to be found.

In the problems (D) and (D_w), the optimal values of the objective functions are representable in terms of the projections of the vectors on the row space of matrix A:

$$\begin{aligned} \text{opt D} &= b^{\top} u_* = b^{\top}(AA^{\top})^{-1} A(c - v_*) = (\bar{x}^{\|})^{\top}(c - v_*)^{\|}, \\ \text{opt } \mathrm{D}_w(\bar{v}) &= b^{\top} w_* = b^{\top}(AA^{\top})^{-1} A(\bar{v} - v_*) = (\bar{x}^{\|})^{\top}(\bar{v} - v_*)^{\|}. \end{aligned}$$

Similarly, in the problem (D_v), the optimal value of the objective function is presented in terms of the projections on the null space of matrix A:

$$\text{opt } \mathrm{D}_v(\bar{x}) = d^{\top} y_* = d^{\top}(K^{\top})^{+}(\bar{x} - x_*) = (\bar{v}^{\perp})^{\top}(\bar{x} - x_*)^{\perp}.$$

Let us consider two cases of LP problems in which the search for solutions is essentially simplified.

Case 1. Let $c^{\perp} = 0_n$; i.e., vector c belongs to the row space of matrix A. Then, according to (3.4), we have $v_*^{\perp} = \tilde{v} = \tilde{v}^{\perp} = \bar{v}^{\perp} = 0_n$, $\bar{v} \in \mathrm{im}\ A^{\top}$, and $d = 0_{\nu}$. The following representation holds:

$$\begin{aligned} \text{opt P} &= c^{\top} x_* = (c^{\|})^{\top} x_*^{\|} = (c^{\|})^{\top} \bar{x}^{\|} = c^{\top} \bar{x}^{\|} = c^{\top} \bar{x}, \\ \text{opt } \mathrm{P}_x(\bar{v}) &= \bar{v}^{\top} x_* = (\bar{v}^{\|})^{\top} x_*^{\|} = (\bar{v}^{\|})^{\top} \bar{x}^{\|} = \bar{v}^{\top} \bar{x}^{\|} = \bar{v}^{\top} \bar{x}. \end{aligned}$$

This means that the objective functions of the problems (P) and (P_x) have the same value for any $x \in \bar{X}$. Taking into account the condition of nonnegativeness of the optimal vector, we obtain that $X_* = \bar{X} \cap \mathbb{R}^n_+$; i.e., any vector $x \in \bar{X}$ with nonnegative components belongs to X_*. From $d = 0_{\nu}$, it follows that the feasible set Y of the problem (P_y) coincides with the solution set Y_* and, in accordance with the optimality conditions (P_y&D_w), the problem (D_w) is reduced to solving the system $A^{\top} w \leq \bar{v}$, $b^{\top} w = \bar{x}^{\top} \bar{v}$. When W_* is found, the set V_* is determined by the first formula in (2.21).

Case 2. Let $\bar{x}^{\|} = 0_n$; i.e., vector $\bar{x}$ belongs to the null space of the matrix A. Then, $b = 0_m$, $x_*^{\|} = \tilde{x} = \bar{x} = \bar{x}^{\|} = 0_n$, opt $\mathrm{D}_v(\bar{x}) = \bar{x}^{\top} v_* = (\bar{x}^{\perp})^{\top} v_*^{\perp} = (\bar{x}^{\perp})^{\top} \bar{v}^{\perp} = \bar{x}^{\top} \bar{v}^{\perp} = \bar{x}^{\top} \bar{v}$, $V_* = \bar{V} \cap \mathbb{R}^n_+$ and $W_* = W$. Hence, the search for the points $v_* \in V_*$ is reduced to finding nonnegative solutions for the system $Kv = d$, which is equivalent to solving the system of linear inequalities $A^{\top} w \leq \bar{v}$.

So, in these two cases the solution of any LP problems is reduced to the solution of a linear system of dimension smaller than original systems (P_x&D_v)–(P_y&D_{vw}).

4. FINDING NORMAL SOLUTIONS

As an example, let us use the nontraditional optimality condition (P_x&D_v) for solving LP problems. We will find the normal solution of this system. Consider the following quadratic programming problem:

$$\min_{x \in \mathbb{R}^n_+} \min_{v \in \mathbb{R}^n_+} \left[(\|x\|^2 + \|v\|^2)/2 : Ax = b,\ Kv = d,\ \bar{v}^{\top} x + \bar{x}^{\top} v = \bar{x}^{\top} \bar{v} \right]; \tag{4.1}$$

i.e., we find the unique 2-norm projection of the origin in $\mathbb{R}^{2n}$ on the nonempty feasible region determined by optimality conditions (P_x&D_v). Similar problems of searching for normal solutions of an LP problem were considered by O. Mangasarian [8, 9]. In problem (4.1) we have $2n$ unknowns, $n+1$ equality constraints, and $2n$ inequalities. In order to simplify this problem, we can introduce a problem dual to (4.1). For this purpose we define the Lagrange function

$$L(x,v,p,q,\alpha) = \left[\|x\|^2 + \|v\|^2\right]/2 + p^\top(b - Ax) + q^\top(d - Kv) + \alpha(\bar{v}^\top x + \bar{x}^\top v - \bar{x}^\top \bar{v}), \quad (4.2)$$

where $p \in \mathbb{R}^m$, $q \in \mathbb{R}^\nu$, $\alpha \in \mathbb{R}^1$ are Lagrange multipliers.

Next we find a saddle point of Lagrange function $L(x,v,p,q,\alpha)$, solving the following problem:

$$\max_{p\,\in\,\mathbb{R}^n} \quad \max_{q\,\in\,\mathbb{R}^\nu} \quad \max_{\alpha\,\in\,\mathbb{R}^1} \quad \min_{x\,\in\,\mathbb{R}^n_+} \quad \min_{v\,\in\,\mathbb{R}^n_+} \quad L(x,v,p,q,\alpha). \quad (4.3)$$

Here, the interior minimization problem will be solved analytically; the exterior maximization problem will be reduced to unconstrained maximization of a concave quadratic function. Necessary and sufficient minimum conditions of the Lagrange function on the set $x \in \mathbb{R}^n_+$, $v \in \mathbb{R}^n_+$ are the following:

$$\begin{aligned} &L_x(x,v,p,q,\alpha) = x - A^\top p + \alpha\bar{v} \geq 0_n, \qquad x \geq 0_n, \qquad D(x)L_x = 0_n,\\ &L_v(x,v,p,q,\alpha) = v - K^\top q + \alpha\bar{x} \geq 0_n, \qquad v \geq 0_n, \qquad D(v)L_v = 0_n. \end{aligned}$$

Solving these relations with respect to x and v, we obtain

$$x = (A^\top p - \alpha\bar{v})_+, \qquad v = (K^\top q - \alpha\bar{x})_+. \quad (4.4)$$

Here and below, $(a)_+$ denotes the vector from $\mathbb{R}^n$ with components $a^i_+ = \max[a^i, 0]$, $i = 1,\ldots,n$, where a^i is the i-component of vector a.

Substituting solutions (4.4) in (4.2), we have

$$\tilde{L}(p,q,\alpha) = b^\top p + d^\top q - \alpha\bar{x}^\top\bar{v} - \left[\|(A^\top p - \alpha\bar{v})_+\|^2 + \|K^\top q - \alpha\bar{x})_+\|^2\right]/2.$$

The problem dual to (4.1) is the following unconstrained maximization problem:

$$\max_{p\,\in\,\mathbb{R}^m} \quad \max_{q\,\in\,\mathbb{R}^\nu} \quad \max_{\alpha\,\in\,\mathbb{R}^1} \quad \tilde{L}(p,q,\alpha). \quad (4.5)$$

If $[x,v,p,q,\alpha]$ is a solution of (4.3), then $[x,v]$ is a solution of (4.1) and $[p,q,\alpha]$ is a solution of (4.5). For the considered quadratic programming problem (4.1), the following converse property holds:

If $[p,q,\alpha]$ is a solution of (4.5), then after its substitution in the formulas (4.4), we receive a solution $[x,v,p,q,\alpha]$ of problem (4.3) and solution $[x,v]$ of problem (4.1). Hence, we obtain the following result.

Theorem 7. *The problems* (P_x), (D_v) *are solvable if and only if the unconstrained maximization problem* (4.5) *is solvable. For each solution* $[p,q,\alpha]$ *of problem* (4.5)*, the vectors*

$$\tilde{x}_* = (A^\top p - \alpha\bar{v})_+, \qquad \tilde{v}_* = (K^\top q - \alpha\bar{x})_+$$

define the unique solution of (4.1)*. Moreover, vectors* x_* *and* v_* *are normal solutions of the corresponding problems* (P_x) *and* (D_v).

Let us mention two properties of problem (4.1) and its dual (4.5).

1. Variables of the primal problem are not included in the formulation of its dual.

2. The dual problem (4.5) is a problem of unconstrained maximization of a differentiable concave piecewise quadratic function in a space of smaller dimension.

Therefore, instead of solving (4.1) we solve problem (4.5) and obtain the solution of the original problem by using formulas (4.4).

Thus, the solution of constrained optimization problem (4.1) with $2n$ unknowns reduces to the unconstrained maximization of a piecewise quadratic differentiable function of $n + 1$ variables. In contrast to the classical exterior penalty function method applied to the problem (P_x), there are no penalty coefficients tending to infinity.

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