Pergamon

Comput. Maths. Math. Phys., Vol.36, No.7, pp. 847–859, 1996

PII: S0965-5425(96)00105-6

DUAL BARRIER-PROJECTION AND BARRIER-NEWTON METHODS FOR LINEAR PROGRAMMING[1]

YU.G. EVTUSHENKO and V.G. ZHADAN

Moscow

(Revised 28 March 2002)

The dual problem of linear programming is considered. Barrier-projection and barrier-Newton methods are proposed for solving it. The convergence of continuous and discrete versions of the methods is proved and estimates of the rate of convergence are given.

INTRODUCTION

Much attention has recently been paid to interior point methods in linear programming, especially after the publication of the paper by Barnes [1]. Whole classes of methods have been developed based on the ideas of projection, scaling and a central path. These methods are surveyed in detail in [2]. Most projective interior point methods, such as Karmarkar's method or the affine scaling method (ASM) (see [3]–[8], for example), are intended for linear programming problems in the standard or a specific canonical form. Versions of the ASM for solving a dual problem in which the admissible set is defined by inequality-type constraints are also considered in [8, 9].

Over the course of a number of years we have developed a different approach to the construction of numerical methods with the properties of interior point methods. It is based on the use of surjective mappings and enables such familiar non-linear programming methods as the gradient projection method or Newton's method to be adapted to solve problems of convex and general non-linear programming, in which the constraints include sets of a "simple structure". These were called barrier-projection and barrier-Newton methods, respectively, in [13]. Versions for linear programming are given in [14]. If the surjective transformation and initial approximations are specially chosen, the barrier-projection method of [14] is the same as the ASM of [3].

Our purpose here is to use this approach to solve the dual problem of linear programming and thereby develop a family of dual barrier-projection and barrier-Newton methods. We will use surjective transformations in order to relax the requirement for the additional dual variables to be non-negative.

The basic idea of the algorithms is explained in Section 1, in which a dual barrier-projection method is constructed on the basis of the stable gradient projection method [16]. Continuous

[1] *Zh. Vychisl. Mat. Mat. Fiz.* Vol.36, No.7, pp. 30–45, 1996.

Financial support for this research was provided by the Russian Foundation for Basic Research (Project No. 96-01-01047).

and discrete versions of the method are investigated, and the influence of the form of transformation on local convergence is determined.

In Section 2 we consider two other barrier-projection methods obtained using other representations of the dual problem from those in Section 1. The ASM of [9] is shown to be a special case of these.

In Section 3 we investigate the global convergence of one of the methods with a special choice of descent step. Finally, in Section 4 we describe a dual barrier-Newton method. The convergence of the methods is proved using Lyapunov's theory of stability.

1. A STABLE VERSION OF THE DUAL METHOD

Consider the linear programming problem

$$\min_{x \in X} c^\top x, \qquad X = \{x \in \mathbb{R}^n : Ax = b, \;\; x \geq 0_n\}. \tag{1.1}$$

Here c is an n-dimensional vector, b is an m-dimensional vector, A is a $m \times n$ matrix of full rank in which $m < n$ and the symbol 0_n denotes the null n-dimensional vector. The columns of A are m-dimensional vectors a_i, $i \in \{1, 2, \dots, n\}$.

The dual problem to (1.1) is the problem

$$\max_{u \in U} b^\top u, \qquad U = \{u \in \mathbb{R}^m : v = c - A^\top u \geq 0_n\}. \tag{1.2}$$

It will be assumed everywhere below that solutions of both problems (1.1) and (1.2) exist and that the set $U_0 = \{u \in \mathbb{R}^m : v = c - A^\top u > 0_n\}$ is non-empty.

The inequality-type constrains in problem (1.2) can be removed by a technique based on the surjective transformation of spaces. This was used for problem (1.1) in [14]. We consider a continuously differentiable n-dimensional vector-function $\varphi(w)$ defined on $\mathbb{R}^n$ for which the closure of the image of the entire space $\mathbb{R}^n$ coincides with the non-negative orthant $\mathbb{R}^n_+$. For simplicity, we shall assume that this function has componential form

$$\varphi(w) = [\varphi^1(w^1), \dots, \varphi^n(w^n)]^\top.$$

Let the function $w^i = \psi^i(v^i)$ be the inverse of $\varphi^i(w^i)$. It exists at least at points $v^i = \varphi^i(w^i)$ where $\dot{\varphi}^i(w^i) \neq 0$. Let

$$\theta(v) = [\theta^1(v^1), \dots, \theta^n(v^n)]^\top, \qquad G(v) = D(\theta(v)),$$

where

$$\theta^i(v^i) = (\gamma^i(v^i))^2, \quad \gamma^i(v^i) = \dot{\varphi}^i(\psi^i(v^i)), \qquad 1 \leq i \leq n,$$

$D(y)$ is a diagonal matrix with ith diagonal element equal to y^i.

We impose two conditions on the transformation $\varphi(w)$.

Condition 1. *The functions $\theta^i(v^i)$, $1 \leq i \leq n$, are defined and continuous in some neighborhood $\mathbb{R}^1_+$ and $\theta^i(v^i) = 0$ if and only if $v^i = 0$.*

Condition 2. *The functions $\theta^i(v^i)$, $1 \leq i \leq n$, are continuously differentiable in some neighborhood $\mathbb{R}^1_+$ and $\dot{\theta}^i(0) > 0$.*

The simplest examples of a transformation $\varphi(w)$ with corresponding functions $\theta(v)$ and $G(v)$ are:

$$\varphi(w) = \frac{1}{4} D(w) w, \quad \theta(v) = v, \qquad G(v) = D(v), \tag{1.3}$$

$$\varphi(w) = e^{-w}, \qquad \theta(v) = D(v)v, \quad G(v) = D^2(v). \tag{1.4}$$

The symbol e^{-w} here denotes a vector-function with components e^{-w^i}, $1 \le i \le n$. Condition 1 is satisfied by both transformations (1.3) and (1.4), and Condition 2 by only the first of these.

Using the transformation $\varphi(w)$, problem (1.2) can be reduced to the following:

$$\max b^\top u, \tag{1.5}$$
$$\varphi(w) - c + A^\top u = 0_n. \tag{1.6}$$

We will use a stable version of the gradient projection method of [16] to solve this problem. Denoting the Lagrange function for problem (1.5), (1.6) by

$$\tilde{L}(u, w, x) = b^\top u - x^\top(\varphi(w) - c + A^\top u)$$

we arrive at the system of ordinary differential equations

$$\frac{du}{dt} = \tilde{L}_u(u, w, x(u, w)), \qquad \frac{dw}{dt} = \tilde{L}_w(u, w, x(u, w)), \tag{1.7}$$

in which the dependence $x(u, w)$ is found by solving the system of linear equations:

$$\tilde{L}_{xu}(u, w, x)\dot{u} + \tilde{L}_{xw}(u, w, x)\dot{w} = -\tau \tilde{L}_x(u, w, x), \quad \tau > 0. \tag{1.8}$$

Since $v = \varphi_w \dot{w}$, in the space of vectors $z = [u, v] \in \mathbb{R}^{m+n}$ the method (1.7), (1.8) takes the form

$$\frac{du}{dt} = b - Ax(z), \quad \frac{dv}{dt} = -G(v)x(z), \tag{1.9}$$
$$\Phi(v)x(z) = A^\top b + \tau(v + A^\top u - c), \tag{1.10}$$

where $\Phi(v) = G(v) + A^\top A$. We put $v(u) = c - A^\top u$.

Lemma 1. *Let the transformation $\varphi(w)$ satisfy Condition 1. Then $\Phi(v(u))$ is a non-singular matrix for any $u \in U_0$.*

Proof. By Condition 1, the matrix $G(v(u))$ is positive definite on U_0. The matrix $A^\top A$ is a Gram matrix and is, therefore, non-negative definite. Thus the entire matrix $\Phi(v(u))$ is positive definite. $\square$

Lemma 2. *Let the assumptions of the previous lemma be satisfied. Furthermore, suppose that the point $u \in U$ can be represented in the form*

$$u = \sum_{j=1}^{s} \alpha_j u_j, \qquad \alpha_j > 0, \quad 1 \le j \le s, \quad \sum_{j=1}^{s} \alpha_j = 1, \tag{1.11}$$

where u_j, $1 \le j \le s$, are the corners of the set U. Then if at least one point u_j is non-degenerate, $\Phi(v(u))$ is a non-singular matrix.

Proof. The matrix $\Phi(v)$, where $v = v(u)$, will be non-singular if it can be shown that the equation

$$\Phi(v)\bar{x} = G(v)\bar{x} + A^\top A\bar{x} = 0_n \tag{1.12}$$

is satisfied if and only if $\bar{x} = 0_n$.

In fact, multiplying (1.12) on the left by $\bar{x}^\top$, we obtain

$$\bar{x}^\top G(v)\bar{x} + \bar{x}^\top A^\top A\bar{x} = 0. \tag{1.13}$$

Since both terms in (1.13) are non-negative, we must have

$$\bar{x}^\top G(v)\bar{x} = 0, \qquad \bar{x}^\top A^\top A\bar{x} = 0. \tag{1.14}$$

We use the notation

$$S_j = \{1 \le i \le n : \alpha_j^\top u_j = c^i\}, \qquad S = \bigcap_{j=1}^{s} S_j.$$

If $S = 0$, then $v > 0_n$ and from the first equation of (1.14) we obtain $\bar{x} = 0_n$. We will now consider the case where $S \neq 0$. Without loss of generality, it can be assumed that $S = \{1, 2, \ldots, k\}$. Let B be the submatrix of the matrix A composed of the first k columns of A, and let N be the submatrix A composed of the remaining $n-k$ columns. Corresponding to this partitioning of A we will also represent the vectors $\bar{x}$ and v as $\bar{x} = [\bar{x}_B, \bar{x}_N]$, $v = [v_B, v_N]$. Since at least one corner u_j is non-degenerate, $k \le m$ and the matrix B has full rank. In addition, according to (1.23) $v_B = 0_k$, $v_N > 0_{n-k}$. It then follows from the first equation of (1.14) that $\bar{x}_N = 0_{n-k}$. Thus the second equation of (1.14) reduces to $\bar{x}_B^\top B^\top B\bar{x}_B = 0$. But this means that $B\bar{x}_B = 0_m$. Since the matrix B is of full rank, it follows that $\bar{x}_B = 0_k$ and, therefore, all the components of the vector $\bar{x}$ are zero. $\square$

Corollary 1. *If the corner u of the set U is non-degenerate, $\Phi(v(u))$ is a non-singular matrix.*

Corollary 2. *Let all the corners of the bounded set U be non-degenerate. Then $\Phi(v(u))$ is a non-singular matrix for any $u \in U$.*

We will introduce the sets

$$W = \{v \in \mathbb{R}^n : v = v(u),\ \ u \in \mathbb{R}^m\}, \qquad V = \{v \in \mathbb{R}^n : v = v(u),\ \ u \in U\}. \tag{1.15}$$

If the set U is a convex polytope with only non-degenerate corners, by Corollary 2 the matrix $\Phi(v)$ has an inverse when $v \in V$. By virtue of continuity, it will also have an inverse in some neighborhood V. For points v of this neighborhood we have

$$x(u,v) = [G(v) + A^\top A]^{-1}[A^\top b + \tau(v + A^\top u - c)]. \tag{1.16}$$

Substituting (1.16) into (1.9), we obtain a different representation of method (1.9), (1.10):

$$\begin{aligned} \frac{du}{dt} &= b - A[G(v) + A^\top A]^{-1}[A^\top b + \tau(v + A^\top u - c)], \\ \frac{dv}{dt} &= -G(v)[G(v) + A^\top A]^{-1}[A^\top b + \tau(v + A^\top u - c)]. \end{aligned}$$

Let $[u(t,z_0), v(t,z_0)]$ be a solution of system (1.9) which satisfies the initial condition $u(t,z_0) = u_0$, $v(t,z_0) = v_0$, $z_0^\top = [u_0^\top, v_0^\top]$. Put $y(u,v) = c - A^\top u - v$. Condition (1.10) can be written in the form

$$\frac{dy(u,v)}{dt} = y_u^\top(u,v)\dot{u} + y_v^\top(u,v)\dot{v} = -\tau y.$$

It follows that system (1.9) has first integral

$$c - A^\top u(t,z_0) - v(t,z_0) = (c - A^\top u_0 - v_0)e^{-\tau t}. \tag{1.17}$$

Thus, $c = A^\top u(t,z_0) - v(t,z_0) \to 0_n$ as $t \to +\infty$. Also, along the paths of the system, according to (1.10),

$$\begin{aligned} b^\top \frac{du}{dt} &= b^\top(b - Ax(z)) = \|b - Ax(z)\|^2 + x^\top(z)A^\top(b - Ax(z)) = \\ &= \|b - Ax(z)\|^2 + x^\top(z)G(v)x(z) + \tau x^\top(z)(c - A^\top u - v). \end{aligned} \tag{1.18}$$

It follows from the second equation of (1.9) that if the transformation $\varphi(w)$ satisfies Condition 1, then neither component of the vector $v(t,z_0)$ changes sign. Thus if $v_0 > 0$, along the entire path $v(t,z_0) > 0$. Thus, since $y(u(t,z_0), v(t,z_0)) \equiv 0_n$ for $y(u_0,v_0) = 0_n$, if $u_0 \in U$ the equation for v can be dropped, which simplifies system (1.9). Instead of (1.9), (1.10) we obtain

$$\frac{du}{dt} = b - Ax(u), \tag{1.19}$$

$$[G(v(u)) + A^\top A]x(u) = A^\top b, \tag{1.20}$$

where $u(0,u_0) = u_0 \in U$. Instead of (1.18) for this system we have the formula

$$b^\top \frac{du}{dt} = \|b - Ax(u)\|^2 + x^\top(u)G(v(u))x(u) \geq 0,$$

that is, the objective function of the dual problem (1.2) increases monotonically on the admissible set. Method (1.19), (1.20) was first proposed in 1977 in [11].

Using Euler's method to integrate system (1.9), (1.10), we obtain

$$u_{k+1} = u_k + \alpha_k(b - Ax_k), \quad v_{k+1} = v_k - \alpha_k G(v_k)x_k, \tag{1.21}$$

$$[G(v_k) + A^\top A]x_k = A^\top b + \tau(v_k + A^\top u_k - c). \tag{1.22}$$

Correspondingly, for system (1.19), (1.20) we have

$$u_{k+1} = u_k + \alpha_k(b - Ax_k), \quad [G(v_k) + A^\top A]x_k = A^\top b, \quad v_k = v(u_k). \tag{1.23}$$

Both these versions of the method solve the direct and dual problems (1.1), (1.2) simultaneously.

Theorem 1. *Let x_* and u_* be non-degenerate solutions of problems* (1.1), (1.2), *respectively, and $v_* = v_*(u_*)$. In addition, let the transformation $\varphi(w)$ satisfy Conditions* 1 *and* 2. *Then:*

(a) the point $z_*^\top = [u_*^\top, v_*^\top]$ is an asymptotically stable position of equilibrium for system (1.9), (1.10);

(b) the solutions $u(t,z_0)$, $v(t,z_0)$ of system (1.9), (1.10) *converge locally exponentially to the point z_*, and the corresponding function $x(z(t,z_0))$ converges to x_*;*

(c) $\alpha_* > 0$ exists such that for any fixed $0 < \alpha_k < \alpha_*$, the sequence $\{[u_k, v_k]\}$ generated by process (1.21), (1.22) *converges locally to z_* at a linear rate; the corresponding sequence $\{x_k\}$ converges to x_*;*

(d) the solutions $u(t,u_0)$ of system (1.19), (1.20) *converge locally exponentially to u_* on U, and the corresponding function $x(u(t,u_0))$ converges to x_*;*

(e) $\alpha_* > 0$ exists such that for any fixed $0 < \alpha_k < \alpha_*$, the sequence $\{u_k\}$ generated by the process (1.23) *converges locally to u_* on U at a linear rate; the corresponding sequence $\{x_k\}$ converges to x_*.*

Proof. We will form an equation in variations for system (1.9), (1.10):

$$\begin{aligned}
\delta\dot{u} &= -A\left[\frac{\partial x(z_*)}{\partial u}\delta u + \frac{\partial x(z_*)}{\partial v}\delta v\right], \\
\delta\dot{v} &= \pm\left\{G(v_*)\frac{\partial x(z_*)}{\partial u}\delta u + \left[D(\dot{\theta}(v_*))D(x_*) + G(v_*)\frac{\partial x(z_*)}{\partial v}\right]\delta v\right\},
\end{aligned}$$

or in matrix form, introducing the notation $\delta z^\top = [\delta u^\top, \delta v^\top]$:

$$\delta\dot{z} = -Q(z_*)\delta z.$$

Here

$$Q(z_*) = \begin{bmatrix} \tau A\Phi_*^{-1}A^\top & A\Phi_*^{-1}[\tau I_n - D(\dot{\theta}(v_*))D(x_*)] \\ \tau G(v_*)\Phi_*^{-1}A^\top & [I_n - G(v_*)\Phi_*^{-1}]D(\dot{\theta}(v_*))D(x_*) + \tau G(v_*)\Phi_*^{-1} \end{bmatrix}, \tag{1.24}$$

where $\Phi_* = G(v_*) + A^\top A$ and I_n is the identity matrix of order n. In evaluating the matrix (1.24) we have used the relations which follow from (1.10):

$$\begin{aligned}
&[G(v) + A^\top A]\frac{\partial x(z)}{\partial u} = \tau A^\top, \\
&D(\dot{\theta}(v))D(x) + [G(v) + A^\top A]\frac{\partial x(z)}{\partial v} = \tau I_n.
\end{aligned}$$

Suppose, to fix our ideas, that a basis of the point x_* consists of the first m columns of matrix A. Then the vectors x_*, v_* and matrices A and $G(v_*)$ have the representations

$$x_* = \begin{bmatrix} x_*^B \\ x_*^N \end{bmatrix}, \qquad v_* = \begin{bmatrix} v_*^B \\ v_*^N \end{bmatrix}, \qquad A = [B\ N], \qquad G(v_*) = \begin{bmatrix} 0_{mm} & 0_{md} \\ 0_{dm} & G_N \end{bmatrix},$$

where $x_*^B > 0_m$, $v_*^B = 0_m$, $x_*^N = 0_d$, $v_*^N > 0_d$, $d = n - m$, $G_N = D(\theta(v_*^N))$ is the right-hand bottom square submatrix of $G(v_*)$ of order d, and 0_{ks} is the $k \times s$ zero matrix.

Since, under the given assumption,

$$\Phi_* = \begin{bmatrix} B^\top B & B^\top N \\ N^\top B & G_N + N^\top N \end{bmatrix},$$

from Frobenius's formula we obtain

$$\Phi_*^{-1} = \begin{bmatrix} B^{-1}[I_m + NG_N^{-1}N^\top](B^\top)^{-1} & -B^{-1}NG_N^{-1} \\ -G_N^{-1}N^\top(B^\top)^{-1} & G_N^{-1} \end{bmatrix}.$$

It follows that

$$\Phi_*^{-1}A^\top = \begin{bmatrix} B^{-1} \\ 0_{dm} \end{bmatrix}.$$

Using this relation, the matrix Q can be reduced to the form

$$Q = \begin{bmatrix} \tau I_m & Q_2 \\ 0_{nm} & Q_1 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} D(\dot{\theta}(v_*^B))D(x_*^B) & 0_{md} \\ Q_3 & \tau I_d \end{bmatrix}, \tag{1.25}$$

where the form of the matrices Q_2 and Q_1 is unimportant.

It follows from (1.25) that the matrix Q has n eigenvalues equal to τ, and m eigenvalues $\dot\theta^i(0)x^i_*$, $1 \le i \le m$. Since the transformation $\varphi(w)$ satisfies Condition 2, these are all strictly positive. Thus, by Lyapunov's theorem on stability for the first approximation, the position of equilibrium (the point z_*) is asymptotically stable, and the solutions of system (1.9), (1.10) converge locally exponentially to z_*.

The convergence of the discrete version (1.21), (1.22) for sufficiently small constants α_k follows from Theorem 2.3.7 of [12].

Since the solutions of system (1.19), (1.20) for $u_0 \in U$ are the same as the corresponding solutions of the more general system (1.9), (1.10) with $v_0 = v(u_0)$, the solutions of (1.19), (1.20) converge locally exponentially to u_* on U. For the same reason, the sequence $\{u_k\}$ generated by the process (1.23) converges locally to u_* on U. This proves the theorem. □

Let η^* and η_* denote the largest and smallest eigenvalues of the matrix Q:

$$\eta^* = \max\left[\tau, \max_{1\le i\le m} \dot\theta^i(0)x^i_*\right], \qquad \eta_* = \min\left[\tau, \max_{1\le i\le m} \dot\theta^i(0)x^i_*\right].$$

By standard arguments (see [12]) it can be shown that the quantity in (c) $\alpha_* = 2/\eta^*$. The rate of convergence will be greatest when the steps in (1.21), (1.22) are equal: $\alpha_k = 2/(\eta_*+\eta^*)$. Then the condition $(\|z_k - z_*\|) \le \varepsilon$ will be satisfied by performing $\ln(\varepsilon/\|z_* - z_0\|)/\ln q$ iterations, where $q = (\eta^* - \eta_*)/(\eta^* + \eta_*)$.

To determine the right-hand side in system (1.19), we need to find the vector $x(u)$ and, therefore, solve a system of n linear equations. If $u \in U_0$, the matrix $G(v(u))$ is non-degenerate and the Sherman–Morrison–Woodberry formula for the inverse of a matrix can be used:

$$[G(v) + A^\top A]^{-1} = G^{-1}(v)\left[I_n - A^\top(I_m + AG^{-1}(v)A^\top)^{-1}AG^{-1}(v)\right].$$

This leads to the system

$$\frac{du}{dt} = [I_m + AG^{-1}(v(u))A^\top]^{-1}b, \qquad u(t,0) = u_0 \in U_0. \tag{1.26}$$

The local convergence to the solution of problem (1.2) on U_0 of method (1.26) and its discrete version again follows from the more general assertions (b) and (d) of Theorem 1.

We note also that in (1.9), (1.10) condition (1.10) could be replaced by any other condition ensuring that the components of the vector $y(u,v)$ decrease to zero. For example, instead of (1.10) we could take

$$[G(v) + A^\top A]x(z) = A^\top b + \tau D(v + A^\top u - c)(v + A^\top u - c), \qquad \tau > 0.$$

Then instead of (1.17), system (1.9), (1.10) would have the first integral

$$c - A^\top u(t,z_0) - v(t,z_0) = D^{-1}(c - A^\top u_0 - v_0 + \tau t)(c - A^\top u_0 - v_0).$$

The statement of Theorem 1 would remain unchanged.

2. OTHER VERSIONS OF DUAL BARRIER-PROJECTION METHODS

By hypothesis, the rank of the matrix A is equal to m and its null-space has dimension $d = n - m$. Let P be a matrix of full rank such that $AP^\top = 0_{md}$. Since the rows of the matrix P are linearly independent, they form a basis in the null-space of the matrix A. If A can be represented in partitioned form $A = [BN]$, where the square matrix B is non-degenerate, we can take P, for example, as the matrix

$$P = \left[\ -N(B^\top)^{-1} \mid I_d\ \right]$$

The definitions (1.15) of the sets W and V can be rewritten using the matrix P in the form

$$W = \{v \in \mathbb{R}^n : P(v - c) = 0_d\}, \qquad V = \{v \in \mathbb{R}^n_+ : P(v - c) = 0_d\}.$$

Let $\bar{x} \in \mathbb{R}^n$ be any vector satisfying the condition $A\bar{x} = b$. Then

$$\max_{u \in U} b^\top u = \max_{u \in U} \bar{x}^\top A^\top u = \max_{v \in V} \bar{x}^\top (c - v) = \bar{x}^\top c - \min_{v \in V} \bar{x}^\top v.$$

Hence, the solution of the dual problem (1.2) can be replaced by the solution of the equivalent minimization problem

$$\min_{v \in V} \bar{x}^\top v. \tag{2.1}$$

The stable version of the barrier-projection method of [14], applied to (2.1), leads to the formulae

$$\frac{dv}{dt} = -G(v)[\bar{x} - P^\top x(v)], \tag{2.2}$$

$$PG(v)P^\top x(v) = PG(v)\bar{x} + \tau P(c - v). \tag{2.3}$$

At points $v \in \mathbb{R}^n$ where the matrix $PG(v)P^\top$ is non-singular, by solving (2.3) we obtain

$$x(v) = [PG(v)P^\top]^{-1}[PG(v)\bar{x} + \tau P(c - v)].$$

Let $H(v) = G^{1/2}(v)$. In addition, consider the right-hand pseudo-inverse matrix $(PH)^+ = (PH)^\top (PGP^\top)^{-1}$ and the projection matrix $(PH)^\# = (PH)^+ PH$. Then method (2.2), (2.3) can be written in projective form as follows:

$$\frac{dv}{dt} = H[\tau(PH)^+ P(c - v) - (I_n - (PH)^\#)H\bar{x}]. \tag{2.4}$$

The first vector in square brackets belongs to the null-space of the matrix AH^{-1}, the second belongs to the row space of the matrix AH^{-1}. We have the formulae

$$P\frac{dv}{dt} = \tau P(c - v), \qquad P(c - v(t, v_0)) = P(c - v_0)e^{-\tau t},$$

from which it is clear that $v(t, v_0)$ approaches the set W as $t \to \infty$.

If $v_0 \in V_0$, where $V_0 = \{v \in V : v > 0_n\}$, then (2.4) possesses the properties of the interior point method; the objective function $x^\top v(v_0, t)$ decreases monotonically and $v(t, v_0) \in V_0$ for all $t \geq 0$. In this case method (2.4) can be rewritten in the form

$$\frac{dv}{dt} = -G(v)[I_n - P^\top (PG(v)P^\top)^{-1} PG(v)]\bar{x}, \qquad v_0 \in V_0. \tag{2.5}$$

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied. Then:*

(a) the point v_* is an asymptotically stable position of equilibrium for system (2.2), (2.3)*;*

(b) the solutions of system (2.2), (2.3) *converge locally exponentially to the point* v_**;*

(c) $\alpha_* > 0$ *exists such that for any constants* $0 < \alpha_k < \alpha_*$*, the discrete version of the method*

$$v_{k+1} = v_k - \alpha_k G(v_k)(\bar{x} - P^\top x_k), \qquad x_k = x(v_k),$$

converges locally to v_* *at a linear rate.*

The proof of this theorem is similar to that of Theorem 1, except that now the matrix Q will have d eigenvalues equal to τ, and m equal to $\dot{\theta}^i(0)x_*^i$, $1 \le i \le m$. The choice of the vector $\bar{x}$ in (2.1), therefore, has no effect on the rate of convergence.

Since $P\dot{v} = 0$ for system (2.5) the vector $\dot{v}$ in this method belongs to the null-space of the matrix P, which is identical with the row space of A. Thus, apart from (2.5), we have

$$\dot{v} = A^\top \lambda \tag{2.6}$$

for some vector $\lambda \in \mathbb{R}^m$. If $v > 0_n$, after multiplying both sides of (2.7) by the matrix $AG^{-1}(v)$ and using the fact that $\dot{v}$ has the form (2.5), we obtain

$$\lambda = -[AG^{-1}(v)A^\top]^{-1}A\bar{x} = -[AG^{-1}(v)A^\top]^{-1}b. \tag{2.7}$$

Substituting (2.7) into (2.6), we arrive at a different representation of the method (2.5):

$$\frac{dv}{dt} = -A^\top[AG^{-1}(v)A^\top]^{-1}b, \qquad v_0 \in V_0. \tag{2.8}$$

In the space of variables u, method (2.8) takes the form

$$\frac{du}{dt} = [AG^{-1}(v(u))A^\top]^{-1}b, \qquad u_0 \in U_0. \tag{2.9}$$

If we use transformations (1.3) and (1.4), from (2.9) we obtain, respectively,

$$\frac{du}{dt} = [AD^{-1}(v(u))A^\top]^{-1}b, \qquad u_0 \in U_0, \tag{2.10}$$

$$\frac{du}{dt} = [AD^{-2}(v(u))A^\top]^{-1}b, \qquad u_0 \in U_0. \tag{2.11}$$

Formula (2.11) is the same as the continuous version of the dual ASM proposed in [9].

It follows from Theorem 2 that the solutions of system (2.2), (2.3) converge locally exponentially to $v_* = v(u_*)$ on the set V. Thus the solutions of system (2.10) also converge locally exponentially to u_* on the set U_0.

If we consider the discrete analogue of method (2.10)

$$u_{k+1} = u_k + \alpha_k[AD^{-1}(v_k)A^\top]^{-1}b, \qquad u_0 \in U_0, \tag{2.12}$$

where $v_k = v(u_k)$, then according to [12] the fact that the continuous version (2.10) has exponential convergence means that the iterative process (2.12) is locally convergent at a linear rate to u_* for sufficiently small constants α_k.

Another version of the dual barrier-projection method can be obtained by representing (1.2) as a problem with $2n$ equality-type constraints:

$$\max b^\top u, \tag{2.13}$$

$$c - A^\top u - v = 0_n, \tag{2.14}$$

$$v - \varphi(w) = 0_n. \tag{2.15}$$

We will put $\tilde{z}^\top = [u^\top, v^\top, w^\top]$ and for (2.13), (2.14), (2.15) construct the Lagrange function

$$\tilde{L}(\tilde{z}, x, y) = b^\top u + x^\top (c - A^\top u - v) + y^\top (v - \varphi(w)).$$

The analogue of method (1.7) for problem (2.13), (2.14), (2.15) is

$$\dot{u} = \tilde{L}_u = b - Ax(\tilde{z}), \qquad \dot{v} = \tilde{L}_v = y(\tilde{z}) - x(\tilde{z}), \qquad \dot{w} = \tilde{L}_w = -\varphi_w^\top(w) y(\tilde{z}), \tag{2.16}$$

where the dependences $x(\tilde{z})$ and $y(\tilde{z})$ are found from the conditions

$$\tilde{L}_{xu}\dot{u} + \tilde{L}_{xv}\dot{v} + \tilde{L}_{xw}\dot{w} = -\tau \tilde{L}_x, \qquad \tilde{L}_{yu}\dot{u} + \tilde{L}_{yv}\dot{v} + \tilde{L}_{yw}\dot{w} = -\tau \tilde{L}_y. \tag{2.17}$$

In variables u, v and $p = \varphi(w)$, system (2.16), (2.17) can be rewritten in the form

$$\frac{du}{dt} = b - Ax(z), \qquad \frac{dv}{dt} = y(z) - x(z), \qquad \frac{dp}{dt} = -G(p)y(z), \tag{2.18}$$

$$(I_n + A^\top A)x(z) - y(z) = A^\top b - \tau(c - A^\top u - v), \tag{2.19}$$

$$(I_n + G(p))y(z) - x(z) = \tau(p - v), \tag{2.20}$$

where $z^\top = [u^\top, v^\top, p^\top]$. Hence, if we take $p_0 = v_0$, we have $p(t, z_0) \equiv v(t, z_0)$ along the entire path $z(t, z_0)$ and it follows from (2.20) that $y(z) = [I_n + G(p)]^{-1} x(z)$. Method (2.18), (2.19), (2.20) is thus simpler in this case: instead of (2.18), (2.19), (2.20) we have

$$\frac{du}{dt} = b - Ax(u, v), \qquad \frac{dv}{dt} = -G(v)[I_n + G(v)]^{-1} x(u, v), \tag{2.21}$$

$$\{G(v)(1 + G(v))^{-1} + A^\top A\} x(u, v) = A^\top b - \tau(c - A^\top u - v). \tag{2.22}$$

Moreover, if it is also assumed that $v_0 = v(u_0)$ then $v(t, z_0) \equiv v(u(t, z_0))$, and using the Sherman–Morrison–Woodberry formula on the set U_0, method (2.21), (2.22) takes the form

$$\frac{du}{dt} = \{I_n + A[I_n + G^{-1}(v(u))]A^\top\}^{-1} b, \qquad u_0 \in U_0. \tag{2.23}$$

The local convergence of method (2.23) and its discrete version

$$u_{k+1} = u_k + \alpha_k \{I_n + A[I_n + G^{-1}(v(u_k))]A^\top\}^{-1} b$$

for the case when $u_0 \in U_0$, $G(v) = D(v)$ and the step α_k is constant and sufficiently small follows from the asymptotic stability of the point $[u_*, v_*, p_*]$, where $p_* = v_* = \varphi(w_*)$ for the more general system (2.18), (2.19), (2.20).

3. THE GLOBAL CONVERGENCE OF THE METHODS

We will consider the global convergence of method (2.12) on the set U_0.

Suppose that problem (1.1) is such that

$$A\bar{e} = 0_m, \tag{3.1}$$

where $\bar{e}^\top = [1, \ldots, 1] \in \mathbb{R}^n$. We will also assume that there is a unique solution u_* of problem (1.2). Then necessarily $C = c^\top \bar{e} > 0$. Let $v_* = v(u_*)$ and $J_*^N = \{i \in \{1, 2, \ldots, n\} : v_*^i > 0\}$. We will consider the Lyapunov function

$$F(u) = \sum_{i \in J_*^N} v_*^i [\ln v_*^i - \ln v^i(u)]. \tag{3.2}$$

The function $F(u)$ is defined, continuously differentiable and non-negative on the set $U_1 = \{u \in \in U : v^i(u) > 0, \;\; i \in J_*^N\}$. Indeed since, according to (3.1),

$$\sum_{i \in J_*^N} v_*^i = \bar{e}^\top v_* = \bar{e}^\top v(u) = \bar{e}^\top c = C > 0, \tag{3.3}$$

it follows from the fact that the arithmetic mean is different from the geometric mean that for any $u \in U_1$

$$F(u) = -C \sum_{i \in J_*^N} \frac{v_*^i}{C} \ln \frac{v^i(u)}{v_*^i} = -C \ln \prod_{i \in J_*^N} \left[\frac{v^i(u)}{v_*^i} \right]^{v_*^i / C} \geq -C \ln \sum_{i \in J_*^N} \frac{v^i(u)}{C} = 0,$$

where equality is only possible when $u = u_*$.

We will compute the derivative of the function (3.2) by virtue of system (2.10). We have

$$\frac{dF(u)}{dt} = F_u^\top \dot{u} = v_*^\top D^{-1}(v(u)) A^\top [A D^{-1}(v(u)) A^\top]^{-1} b. \tag{3.4}$$

We put

$$p(u) = [A D^{-1}(v(u)) A^\top]^{-1} b, \qquad x(u) = D^{-1}(v(u)) A^\top p(u).$$

The vector $x(u)$ thus defined satisfies the equation $Ax(u) = b$. Moreover, according to (3.1), $x^\top(u) v(u) = \bar{e}^\top A^\top p(u) = 0$. Thus

$$x^\top(u) c = x^\top(u)(v(u) + A^\top u) = u^\top A x(u) = b^\top u.$$

From this and (3.4) we obtain

$$\frac{dF(u)}{dt} = v_*^\top x(u) = x^\top(u)(c - A^\top u_*) = b^\top u - b^\top u_* \leq 0, \tag{3.5}$$

where equality is only possible if $u = u_*$.

For any $u_0 \in U_0$, we put $Q = \{u \in U_1 : F(u) \leq F(u_0)\}$. This set is compact because, by (3.3), the set V is compact and, therefore, the set U is also. Moreover, the set Q does not contain any corners of U apart from u_*. It follows from inequality (3.5) that $u(t, u_0) \in Q$ for all $t \geq 0$.

We now put

$$K = \inf_{u \in Q} \frac{\langle b, u_* - u \rangle}{F(u)}. \tag{3.6}$$

On the basis of (3.5) and (3.6) we then have

$$F(u(t, u_0)) \leq F(u_0) e^{-Kt}, \qquad t \geq 0.$$

Lemma 3. *In problem (1.2), let there be a unique non-degenerate solution u_*. Then the quantity K has a lower bound*

$$K \geq \frac{1 - \exp\left[-F(u_0)/C\right]}{F(u_0)} \min_{1 \leq j \leq m} s_j > 0, \tag{3.7}$$

where $s_j = b^\top(u_* - u_j)$ and the vertex u_j is adjacent to u_* in the polytope U, $1 \leq j \leq m$.

Proof. We now introduce variables $z = u - u_*$. In these variables the function $F(u)$ and formula (3.6) for finding the value of K take the form

$$\tilde{F}(z) = -\sum_{i \in J_*^N} v_*^i \ln\left\{1 - \frac{a_i^\top z}{v_i^*}\right\}, \qquad K = -\sup_{z \in Q_1} \frac{\langle b, z\rangle}{\tilde{F}(z)},$$

where $Q_1 = \{z \in Z : \tilde{F}(z) \leq F(u_0)\}$, $Z = \{z \in \mathbb{R}^m : A^\top z \leq v_*\}$. The function $\tilde{F}(z)$ is convex with respect to z on Q_1. We have $\tilde{F}(0) = 0$ and $\tilde{F}(z) > 0$, $\langle b, z\rangle < 0$ for all $z \in Z$, $z \neq 0_m$. Thus for any point $\bar{z} \in S = \{z \in Q_1 : \tilde{F}(z) = F(u_0)\}$ and any $0 < \alpha \leq 1$ we have $\tilde{F}(\alpha\bar{z}) \leq \alpha\tilde{F}(\bar{z})$. It follows that

$$\frac{\langle b, \alpha\bar{z}\rangle}{\tilde{F}(\alpha\bar{z})} \leq \frac{\langle b, \bar{z}\rangle}{\tilde{F}(\bar{z})}, \qquad K = -\frac{1}{F(u_0)} \max_{z \in S} \langle b, z\rangle. \tag{3.8}$$

The point $z = 0$ is a vertex of the polytope Z. Let z_j be other vertices adjacent to that vertex, and let β_j be a solution of the equation

$$\sum_{i \in J_*^N} v_*^i \ln(1 - \beta_j q_{ij}) + F(u_0) = 0, \tag{3.9}$$

where $q_{ij} = a_i^\top z_j / v_*^i$. Since $\tilde{F}(z_j) = +\infty$ we have $0 < \beta_j < 1$. We have

$$\max_{z \in S} \langle b, z\rangle = \max_{1 \leq j \leq n} \beta_j \langle b, z_j\rangle = -\min_{1 \leq j \leq m} \beta_j s_j < 0. \tag{3.10}$$

Since $A^\top z_j \leq v_*$ we have $q_{ij} \leq 1$ for all $i \in J_*^N$, and for at least one index i, we have $q_{ij} = 1$. Therefore,

$$\ln(1 - \beta_j q_{ij}) \geq \ln(1 - \beta_j)$$

and, consequently, any solution β_j of (3.9), $1 \leq j \leq m$, satisfies the inequality $\beta_j \geq \bar{\beta}$, where $\bar{\beta}$ is a solution of the equation

$$\ln(1 - \bar{\beta}) \sum_{i \in J_*^N} v_*^i + F(u_0) = 0.$$

Hence

$$\bar{\beta} = 1 - exp[-F(u_0)/C]. \tag{3.11}$$

From (3.8), (3.10) and (3.11) we obtain the bound (3.7). This proves the lemma. □

We put

$$\mu(u) = \max_{1 \leq i \leq n} x^i(u).$$

The quantity $\mu(u) > 0$ for any $u \in U_0$. For if $\mu(u) \leq 0$, then $x(u) \leq 0_n$, and $x^i(u) < 0$ for at least one index i. Then for any $\alpha > 0$ we have $\alpha x(u) \leq 0_n < \bar{e}$. Multiplying this inequality by the matrix $D(v(u))$, we obtain $\alpha A^\top [AD^{-1}(v)A^\top]^{-1} b \leq v(u)$, or

$$A^\top \{u + \alpha[AD^{-1}(v)A^\top]^{-1} b\} \leq c. \tag{3.12}$$

Thus, the vector $u + \alpha[AD^{-1}(v)A^\top]^{-1}b$ belongs to the set U for any $\alpha > 0$, which contradicts the fact that the set U is bounded. It also follows from (3.12) that the quantity $1/\mu(u)$ is an upper bound for α for which $u + \alpha x(u) \in U$.

Theorem 3. *Let the step α_k of* (2.12) *be chosen from the condition*

$$0 < \alpha_k = \gamma/\mu(u_k), \qquad 0 < \gamma < 1. \tag{3.13}$$

Then for any $u_0 \in U_0$ a $0 < \gamma(u_0) < 1$ exists such that for all $0 < \gamma \le \gamma(u_0)$ and $k \ge 0$,

$$F(u_{k+1}) \le F(u_k)(1 - 0.5\alpha_k K), \tag{3.14}$$

where the quantity K is defined by (3.6).

Proof. We will compute the change of the function (3.2) in one step of the iterative process. We have

$$\begin{aligned} F(u_{k+1}) &= -\sum_{i \in J_*^N} v_*^i \ln\left[1 - \frac{\alpha_i^\top(u_k + \alpha_k p_k - u_*)}{v_i^*}\right] = \\ &= -\sum_{i \in J_*^N} v_*^i \ln\left[\frac{v_k^i}{v_*^i}(1 - \alpha_k x_k)\right] = F(u_k) - \sum_{i \in J_*^N} v_*^i \ln(1 - \alpha_k x_k^i). \end{aligned} \tag{3.15}$$

Here $p_k = p(u_k)$, $x_k = x(u_k)$. Let

$$\Delta(u, \alpha) = \alpha^{-1} \sum_{i \in J_*^N} v_*^i \ln[1 - \alpha^i x(u)], \qquad \alpha > 0. \tag{3.16}$$

Finding a series expansion of the right-hand side of (3.16), we obtain

$$\Delta(u, \alpha) = -v_*^\top x(u) - \frac{\alpha}{2} \sum_{i \in J_*^N} \frac{v_*^i [x^i(u)]^2}{[1 - \alpha\theta^i(u) x^i(u)]^2},$$

where $0 \le \theta^i(u) \le 1$, $i \in J_*^N$. Hence from (3.5) we have

$$\Delta(u, \alpha) \ge b^\top(u_* - u) - \frac{\gamma}{2(1-\gamma)^2 \mu(u)} \sum_{i \in J_*^N} v_*^i [x^i(u)]^2, \tag{3.17}$$

which holds for any $\alpha \le \gamma/\mu(u)$.

We now put

$$r(u) = \mu(u) b^\top (u_* - u) \left[\sum_{i \in J_*^N} v_*^i [x^i(u)]^2\right]^{-1}, \qquad \bar{r} = \inf_{u \in Q} r(u)$$

and show that $\bar{r} > 0$. This inequality in fact applies if the infimum is reached at a point $u \in Q$ different from u_*. In the case when there is a sequence $\{u_s\}$ which converges to the point u_* such that $\bar{r} = \lim_{s \to \infty} r(u_s)$ we will show that again $\bar{r} > 0$.

To fix our ideas, let the point u_* be such that $v_*^\top = [v_*^B, v_*^N]$, where $v_*^B \in \mathbb{R}^m$, $v_*^N \in \mathbb{R}^d$, $v_*^B = 0_m$, $v_*^N > 0_d$. We will use the same partition for an arbitrary vector $v(u)$, and also for a matrix $A = [BN]$. Let $\Gamma^B(u) = BD^{-1}(v^B(u))B^\top$, $\Gamma^N(u) = ND^{-1}(v^\top(u))N^\top$.

Since B is a non-degenerate matrix, for all $u \in U_0$

$$\Gamma(u) = AD^{-1}(v(u))A^\top = \Gamma^B(u) + \Gamma^N(u) = \Gamma^B(u)[I + (\Gamma^B(u))^{-1}\Gamma^N(u)].$$

Thus

$$\Gamma^{-1}(u) = \{I - (\Gamma^B(u))^{-1}\Gamma^N(u) + [(\Gamma^B(u))^{-1}\Gamma^N(u)]^2 - \ldots\}(\Gamma^B(u))^{-1} = (\Gamma^B(u))^{-1} + \Phi(u),$$

where $\|\Phi(u)\| = o(\|u - u_*\|)$. From this we obtain

$$\begin{aligned} x^B(u) &= D^{-1}(v^B)B^\top(\Gamma^B(u))^{-1}b + D^{-1}(v^B)B^\top\Phi(u)b = x_*^B + \varphi_1(u), \\ x^N(u) &= D^{-1}(v^N)N^\top\Gamma^{-1}(u)b = \varphi_2(u), \\ \mu(u) &= \max_{1 \le i \le m} x_*^i + \varphi_3(u), \\ \|\varphi_i(u)\| &= O(\|u - u_*\|), \qquad i = 1, 2, 3. \end{aligned}$$

Let the sequence $u_s \to u_*$, $u_s \in U_0$. If $\bar{r} = 0$, then $r(u_s) < 1$ for sufficiently large s. But since $\|x^N(u)\| = O(\|u - u_*\|)$, we obtain

$$\sum_{i \in J_*^N} v_*^i(x^i(u)) = o(\|u - u_*\|),$$

and, therefore, the inequalities $r(u_s) < 1$ cannot apply for large s. The resulting contradiction shows that $\bar{r} > 0$.

Since $\bar{r} > 0$, there is a sufficiently small $0 < \gamma(u_0) < 1$ such that

$$\gamma(1-\gamma)^{-2}\mu^{-1}(u) \sum_{i \in J_*^N} v_*^i(x^i(u))^2 \leq b^\top(u_* - u)$$

for all $0 < \gamma \leq \gamma(u_0)$ and $u \in Q$. Thus, for these u, γ and $\alpha \leq \gamma/\mu(u)$, according to (3.17), $\Delta(u, \alpha) \geq b^\top(u_* - u)/2$. Hence from (3.15) and the inequality $b^\top(u_* - u) \leq KF(u)$, we obtain the required bound (3.14). This completes the proof of the theorem. $\square$

Now let

$$B(u_0) = \max_{u \in Q(u_0)} \max_{1 \leq j \leq n} x^i(u), \qquad \bar{\alpha}(u_0) = \gamma/B(u_0).$$

Then if the step α_k is chosen from condition (3.13) we have $\alpha_k \geq \bar{\alpha}(u_0)$ for any $k \geq 0$. Thus, in addition to (3.14), we have

$$F(u_{k+1}) \leq F(u_k)(1 - 0.5\alpha K), \tag{3.18}$$

where $0 < \alpha \leq \bar{\alpha}(u_0)$. The number of steps needed before process (2.12) reaches a certain neighborhood of the point u_* can be estimated using (3.18). Inequality (3.18) also applies to process (2.12) with constant step $\alpha_k = \alpha \leq \bar{\alpha}(x_0)$.

4. THE DUAL BARRIER-NEWTON METHOD

If the expression from (1.20) for $x(u)$ is substituted into the admissibility condition, we arrive at the following equation:

$$b - Ax(u) = 0. \tag{4.1}$$

We will solve (4.1) by Newton's method. Its continuous version leads to the system

$$\Lambda(u)\frac{du}{dt} = Ax(u) - b, \tag{4.2}$$

where $\Lambda(u)$ is the total derivative with respect to u of the vector-function $Ax(u)$.

Differentiating the identity (1.20) with respect to u, we obtain

$$-D(\dot{\theta}(v))D(x)A^\top + (D(\theta(v)) + A^\top A)\frac{dx}{du} = 0.$$

Therefore,

$$\Lambda(u) = -A[D(\theta(v(u))) + A^\top A]^{-1} D(\dot{\theta}(v(u)))D(x(u))A^\top,$$

and if this matrix is non-singular, method (4.2) can be rewritten in the form

$$\frac{du}{dt} = [A[D(\theta(v(u))) + A^\top A]^{-1} D(\dot{\theta}(v(u)))D(x(u))A^\top]^{-1}[b - Ax(u)]. \tag{4.3}$$

Lemma 4. *Let the solutions x_* and u_* of the two linear programming problems* (1.1) *and* (1.2) *be non-degenerate, and let Conditions* 1 *and* 2 *hold. Then the matrix $\Lambda(u_*)$ is non-singular.*

Proof. For $v_* = v_*(u_*)$, we have $(D(\theta(v_*)) + A^\top A)x_* = A^\top b$ and, therefore, $x(u_*) = x_*$. Suppose, to fix our ideas, that a basis of the point x_* is formed by the first m columns of matrix A. Then we have the representation $A = [BN]$, where B is a square non-singular matrix. Since in that case

$$A[D(\theta(v_*)) + A^\top A]^{-1} = [(B^\top)^{-1} \mid 0_{md}],$$

we obtain

$$\Lambda(u_*) = (B^\top)^{-1} D(\dot{\theta}(v_*^B) D(x_*^B) B. \tag{4.4}$$

All the square matrices on the right-hand side of (4.4) are non-degenerate, and so the matrix $\Lambda(u_*)$ is non-singular. □

Using Lemma 4, we can formulate a theorem on the local convergence of method (4.3) and its discrete version

$$u_{k+1} = u_k + [A[D(\theta(v_k)) + A^\top A]^{-1} D(\dot{\theta}(v_k)) D(x_k) A^\top]^{-1} (b - Ax_k), \tag{4.5}$$

where $v_k = v(u_k)$, $x_k = x(u_k)$.

Theorem 4. *Let the conditions of Lemma* 4 *be satisfied. Then the point u_* is an asymptotically stable position of equilibrium for system* (4.3). *Moreover, if the matrix $\Lambda(u)$ satisfies a Lipschitz condition in some neighborhood of u_*, the sequence $\{u_k\}$ generated by the process* (4.5) *converges locally to u_* at a quadratic rate.*

The form of (4.3) can be simplified slightly if $\varphi(w)$ is taken as the transformation (1.3):

$$\frac{du}{dt} = [A[D(v(u)) + A^\top A]^{-1} D(x(u)) A^\top]^{-1} [b - Ax(u)].$$

Its discrete version is similar to (4.5).

REFERENCES

1. KARMARKAR N., A new polynomial time algorithm for linear programming. *Combinatorica* **4**, 373–395, 1984.

2. GONZAGA C.C., Path-following algorithms for linear programming. *SIAM Rev.* **34**, 167–224, 1992.

3. DIKIN I.I., Iterative solution of problems of linear and quadratic programming. *Dokl. Akad. Nauk SSSR* **174**, 747–748, 1967.

4. BARNES E., A variation on Karmarkar's algorithm for solving linear programming problems. *Math. Program.* **36**, 174–182, 1986.

5. VANDERBEI R., MEKETON M. and FREEDMAN B., A modification of Karmarkar's linear programming algorithm. *Algorithmica* **1**, 395–407, 1986.

6. BAYER D.A. and LAGARIAS J.C., The nonlinear geometry of linear programming. Affine and projective scaling trajectories. *Trans. Amer. Math. Soc.* **314**, 499–526, 1989.

7. HERZEL S., RECCHIONI M.C. and ZIRILLI F., A quadratically convergent method for linear programming. *Linear Algebra Appl.* **152**, 255–289, 1991.

8. ZORKAL'TSEV V.I., Algorithms of projective optimization which use the multipliers of previous iterations. *Zh. Vychisl. Mat. Mat. Fiz.* **34**, 1095–1103, 1994.

9. ADLER I., KARMARKAR N., RESENDE M.G.C. and VEIGA G., An implementation of Karmarkar's algorithm for linear programming. *Math. Program.* **44**, 297–335, 1989.

10. YEVTUSHENKO Yu.G. and ZHADAN V.G., Numerical solution of some problems in operational research. *Zh. Vychisl. Mat. Mat. Fiz.* **13**, 583–597, 1973.

11. YEVTUSHENKO Yu.G. and ZHADAN V.G., A relaxation method for solving problems of nonlinear programming. *Zh. Vychisl. Mat. Mat. Fiz.* **17**, 890–904, 1977.

12. YEVTUSHENKO Yu.G., *Methods of Solving Extremal Problems and their Application in Optimization Systems.* Nauka, Moscow, 1982.

13. EVTUSHENKO Yu.G. and ZHADAN V.G., Stable barrier-projection and barrier-Newton methods in nonlinear programming. *Optimizat. Methods Software* **3**, 237–256, 1994.

14. EVTUSHENKO Yu.G. and ZHADAN V.G., Stable barrier-projection and barrier-Newton methods in linear programming. *Comput. Optimizat. Appl.* **3**, 289–303, 1994.

15. YEVTUSHENKO Yu.G. and ZHADAN V.G., Barrier-projection methods in non-linear programming. *Zh. Vychisl. Mat. Mat. Fiz.* **34**, 669–684, 1994.

16. TANABE K., A geometric method in nonlinear programming. *J. Optimizat. Theory Appl.* **30**, 181–210, 1980.